Arcs on Determinantal Varieties

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THESIS

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Á miña nai.

To Jing.

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R.D.Á.

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SUMMARY

We study the structure of the arc space and the jet schemes of generic determinantal varieties. Via the use of group actions, we are able to compute the number of irreducible components of all the jet schemes, find formulas for log canonical thresholds, and compute some motivic volumes.

We also study extensions of our results for determinantal varieties to more general quasi-homogeneous spaces, with focus on spherical varieties. We obtain good descriptions for the space of skew-symmetric matrices, and for toroidal embeddings of symmetric spaces.

CHAPTER 1

INTRODUCTION

The purpose of this monograph is to study arc spaces and jet schemes of generic determinantal varieties.

Arcs and jets are higher order analogues of tangent vectors. Given a variety X defined over $\mathbf{C}$, an arc of X is a $\mathbf{C}[\![t]\!]$-valued point of X, and an n-jet is a $\mathbf{C}[t]/(t^{n+1})$-valued point. A 1-jet is the same as a tangent vector. Just as in the case of the tangent space, arcs on X can be identified with the closed points of a scheme X_∞, which we call the *arc space* of X (Nash, 1995; Vojta, 2007). More formally, there is a scheme X_∞ whose functor of points is:

$$X_\infty(A) = \operatorname{Hom}\left(\operatorname{Spec} A[\![t]\!], X\right).$$

Similarly, n-jets give rise to the *n-th jet scheme* of X, which we denote by X_n.

Arc spaces and jet schemes are useful tools in birational geometry and in the study of singularities. Among their applications, the better known is probably the theory of motivic integration (Kontsevich, 1995; Denef and Loeser, 1999); it was introduced by Kontsevich as an analogue of p-adic integration, and gives a framework where one can define topological and geometric invariants on singular varieties that behave nicely under birational transformations. This is done by defining a measure on the arc space with values on a localization of the Grothendieck group of varieties, and studying a change of variables formula for this measure. But there is a more elementary reason explaining the importance of arc spaces in birational geometry: their close relation to valuation theory.

Assume that X is an irreducible complex variety and consider a discrete valuation ν of the function field of X, positive over X. Since the completion of a discrete valuation ring is a power series ring in one variable, ν induces an arc $\operatorname{Spec} k_\nu[\![t]\!] \to X$, where k_ν is the residue field of ν, and we obtain a k_ν-valued point of the arc space $\operatorname{Spec} k_\nu \to X_\infty$. In fact we get infinitely many such points, corresponding to different choices of uniformizing parameter for $k_\nu[\![t]\!]$. All of these points verify that their associated arc $\operatorname{Spec} k_\nu[\![t]\!] \to X$ is dominant, they are what we call *fat* points of the arc space (Ishii, 2008).

Conversely, for an arbitrary k-valued fat point of the arc space $\operatorname{Spec} k \to X_\infty$, one gets an inclusion of fields $\mathbf{C}(X) \subset k(\!(t)\!)$, and the canonical valuation on $k(\!(t)\!)$ induces a valuation on X. In other words, there is a well-defined map

$$\{\text{ fat points of } X_\infty \} \longrightarrow \{\text{ discrete valuations over } X \}.$$

In the previous paragraph we saw that this map is surjective, but far from being injective. On the other hand, if we restrict ourselves to *divisorial* valuations (those that correspond to a prime divisor in some birational model dominating X), there is a natural way to define a section to this map (Ein et al., 2004; Ishii, 2008; de Fernex et al., 2008). This arises from the notion of *maximal divisorial point* associated to a divisorial valuation ν: the unique fat point of X_∞ inducing ν and dominating any other fat point also inducing ν. The existence of such points is can be proven using resolution of singularities. In this way we obtain an inclusion of the set of divisorial valuations into the arc space:

$$\{\text{ divisorial valuations over } X \;\} \xrightarrow{\simeq} \{\text{ maximal divisorial points of } X_\infty \;\} \;\subset\; X_\infty.$$

As a consequence one gets a topology in the set of divisorial valuations.

The first to use the topology of the arc space as a tool to study valuations is Nash, who is also responsible for introducing arc spaces into algebraic geometry (Nash, 1995). He looked at the family of arcs going through the singular locus of the variety and considered its fat irreducible components. Without using this language, he basically showed that these components give maximal divisorial points, and conjectured that these points correspond to the *essential* divisors of X (those that appear in any resolution of singularities of X). This conjecture has been shown to be false in dimensions 4 and higher (Ishii and Kollár, 2003), but is still open in dimensions 2 and 3 (Lejeune-Jalabert, 1990; Nobile, 1991; Lejeune-Jalabert and Reguera, 1998; Plénat, 2005; Plénat and Popescu-Pampu, 2006; Lejeune-Jalabert and Reguera, 2008).

Another remarkable application of this approach is Mustaţă's theorem: when X is smooth, the log discrepancy of a divisorial valuation coincides with the codimension of the corresponding maximal divisorial point in the arc space X_∞ (Mustaţă, 2002; Ein et al., 2004; de Fernex et al., 2008). This allows us to study singularities of pairs using arc spaces.

For example, in the case of local complete intersection varieties, one can show that the notions of terminal, canonical and log canonical are equivalent to the equidimensionality, irreducibility and normality of the jet schemes, respectively (Mustaţă, 2001; Ein et al., 2003; Ein and Mustaţă, 2004). Still in the complete intersection case, arc spaces can also be used to prove a version of Inversion of Adjunction (Ein et al., 2003; Ein and Mustaţă, 2004).

Despite their unquestionable theoretical usefulness, arc spaces and jet schemes are often hard to compute in concrete examples. The main difficulty arises from the structure of their equations, which are obtained by "differentiating" the original equations of the variety (Vojta, 2007), but in a way that increases their complexity. Even in cases where the base is reasonably well understood, one can say very little about the arc spaces or the jet schemes. For example, if we consider a singular toric variety, it is still unknown how to compute the number of irreducible components of the jet schemes.

The topology of the arc space has been studied in depth only for a few classes of varieties. There are some results for surfaces (due to the interest in Nash's conjecture, see references above), quotient singularities (Denef and Loeser, 2002), monomial ideals (Goward and Smith, 2006; Yuen, 2007b), and toric varieties (Ishii, 2004). But beyond these cases very little is known. The purpose of this monograph is to analyze in detail the structure of the arc space and jet schemes of generic determinantal varieties, giving a new family of examples for which the arc space is well understood. Also, we will explore extensions of our results to more general quasi-homogeneous spaces.

Consider affine space $M = \mathbf{A}^{rs}$ of dimension rs, and think of it as the space of matrices of size $r \times s$. Assume $r \leq s$. Inside of M we look at the variety D^k whose points correspond to matrices of rank at most k. The varieties D^k are known as the *generic determinantal varieties* (see (Bruns and Vetter, 1988) for a comprehensive study). These spaces (and their generalizations) appear naturally in many branches of algebraic geometry, notably in the study of moduli spaces or when dealing with problems arising in representation theory. When $0 < k < r$, the variety D^k is singular along D^{k-1}, giving interesting examples of singular algebraic varieties.

Attempts at the study of jet schemes of determinantal varieties have appeared previously in the literature (Košir and Sethuraman, 2005a; Košir and Sethuraman, 2005b; Yuen, 2007a). Up to now, the approach has always been to use techniques from commutative algebra: perform a careful study of the defining ideal, and try to get either a Gröbner basis or a good approximation for it. This has been quite successful for ranks 1 and $r-1$, especially when $r = s$, but the general case seems too complex for these methods. The basic unanswered question is to compute the number of irreducible components of the jet schemes.

Our approach is quite different in nature: we focus on the natural group action. This is a technique already present in Ishii's study of the arc spaces of toric varieties (Ishii, 2004). Consider the group $G = \mathrm{GL}_r \times \mathrm{GL}_s$, which acts on the space of matrices M via change of basis. The rank of a matrix is the unique invariant for this action, the orbit closures being precisely the determinantal varieties D^k. The assignments sending a variety X to its arc space X_∞ and its jet schemes X_n are functorial. Since G is an algebraic group and the action on M is rational, we see that G_∞ and G_n are also groups, and that they act on M_∞ and M_n, respectively. Determinantal varieties are G-invariant, hence their arc spaces are G_∞-invariant and their jet schemes are G_n-invariant. The main observation is that most questions regarding components and dimensions of jet schemes and arc spaces of determinantal varieties can be reduced to the study of orbits in M_∞ and M_n. The development of this idea is the content of Chapter 3.

Orbits in the arc space M_∞ are easy to classify. As a set, M_∞ is just the space of matrices with coefficients in $\mathbf{C}[\![t]\!]$, and G_∞ acts via change of basis over the ring $\mathbf{C}[\![t]\!]$. Gaussian elimination allows us to find representatives for the orbits: each of them contains a unique diagonal matrix of the form $\operatorname{diag}(t^{\lambda_1}, \dots, t^{\lambda_r})$, where $\infty \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 0$, and the sequence $\lambda = (\lambda_1, \dots, \lambda_r)$ determines the orbit. In Section 3.2 we see how to decompose arc spaces and jet schemes of determinantal varieties as unions of these orbits. Once this is done, the main difficulty to determine irreducible components is the understanding of the topology of M_∞. Specifically: which orbits are contained in the closure of a given orbit? This is answered in Section 3.3, where the following theorem is proved.

Theorem. *Consider two sequences $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r \geq 0)$ and $\lambda' = (\lambda'_1 \geq \cdots \geq \lambda'_r \geq 0)$, and let $\mathcal{C}_\lambda$ and $\mathcal{C}_{\lambda'}$ be the corresponding orbits in the arc space M_∞. Then the closure of $\mathcal{C}_\lambda$ contains $\mathcal{C}_{\lambda'}$ if and only if*

$$\lambda_r + \lambda_{r-1} + \cdots + \lambda_{r-k} \quad \leq \quad \lambda'_r + \lambda'_{r-1} + \cdots + \lambda'_{r-k} \qquad \forall k \in \{0, \dots, r\}.$$

This result is proved in Section 3.3 as Theorem 3.3.11. One of its most significant consequences is the determination of the number of irreducible components of the jet schemes of determinantal varieties. This computation is carried out in Section 3.4, where the following result appears as Corollary 3.4.6.

Theorem. *Let D^k_n be the n-th jet scheme of the determinantal variety D^k of $r \times s$ matrices of rank at most k, where $r \leq s$. If $k = 0$ or $k = r - 1$, the jet scheme D^k_n is irreducible. Otherwise the number of irreducible components of D^k_n is*

$$n + 2 - \left\lceil \frac{n+1}{k+1} \right\rceil .$$

As mentioned earlier, one can get log discrepancies for divisorial valuations by computing codimensions of the appropriate sets in the arc space. In the case at hand, the most natural valuations one can look at are the invariant divisorial valuations. In Section 3.5 we see that the maximal divisorial points corresponding to these valuations are precisely the generic points of the orbits in M. Hence computing log discrepancies gets reduced to computing codimensions of orbits. This explains the relevance of the following result, which appears in Section 3.5 as Proposition 3.5.5.

Theorem. *Consider a sequence $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r \geq 0)$ and let $\mathcal{C}_\lambda$ be the corresponding orbit in the arc space M_∞. Then the codimension of $\mathcal{C}_\lambda$ in M_∞ is*

$$\operatorname{codim}(\mathcal{C}_\lambda, M_\infty) = \sum_{i=1}^{r} \lambda_i (s - r + 2i - 1).$$

Once these codimensions are known, one can compute log canonical thresholds for pairs involving determinantal varieties. The following result appears in Section 3.5 as Theorem 3.5.7.

Theorem. *Recall that M denotes the space of matrices of size $r \times s$ and D^k is the variety of matrices of rank at most k. The log canonical threshold of the pair (M, D^k) is*

$$\operatorname{lct}(M, D^k) = \min_{i=0\ldots k} \frac{(r-i)(s-i)}{k+1-i}.$$

We should note that the previous result is not new. Log resolutions for generic determinantal varieties are now classical objects. They are essentially spaces of complete collineations, obtained by blowing up D^k along D^0, D^1, $\ldots$, D^{k-1}, in this order (Semple, 1951; Tyrrell, 1956; Vainsencher, 1984; Laksov, 1987). It is possible to use these resolutions to compute log canonical thresholds, and this was done by Amanda Johnson in her Ph.D. thesis (Johnson, 2003). In fact she is able to compute all the multiplier ideals $\mathcal{J}(M, c \cdot D^k)$.

Using our techniques, we are able to compute more than just codimensions of orbits. As an example of the possibilities of the method, in Section 3.6 we compute motivic volumes of orbits. This could also have been done using spaces of complete collineations, but our approach does not need any knowledge about the structure of such resolutions.

As mentioned above, our method is inspired by Ishii's study of the arc spaces of toric varieties (Ishii, 2004). It turns out that there are many similarities between toric varieties and determinantal varieties. They are both examples of *quasi-homogeneous* spaces: they have an action by an algebraic group which has a dense orbit. Moreover, the action is particularly nice, the dense orbit being what is known as a *spherical homogeneous space* (see Section 4.2). The purpose of Chapter 4 is the exploration of the consequences of this connection for the study of the arc space.

There is a classification theory for embeddings of spherical homogeneous spaces analogous to the classification of toric varieties in terms of fans (Luna and Vust, 1983; Knop, 1991). This theory was initiated in the seminal work of Luna and Vust (Luna and Vust, 1983), where one can already find a discussion about arcs on homogeneous spaces (see Section 4.1). They basically classify orbits in the arc space, but ignore the topology, the containment relation between orbit closures. Surprisingly, if one writes the results about orbit closures in the cases of toric varieties and determinantal varieties in the language of spherical varieties, the analogy is evident. In Section 4.2 we explain this analogy, and then we start to examine the situation for other types spherical varieties. We obtain complete descriptions in the cases of skew-symmetric matrices (Section 4.4) and toroidal embeddings of symmetric spaces (Section 4.5).

CHAPTER 2

ARC SPACES AND MOTIVIC INTEGRATION

We briefly review in this chapter the basic theory of arc spaces and motivic integration, as these tools will be used repeatedly. Most of these results are well-known. We have gather them mainly from (de Fernex et al., 2008), (Denef and Loeser, 1999), (Ein et al., 2004), and (Ishii, 2008). We direct the reader to those papers for more details and proofs.

In this chapter and in the rest of the monograph, we will always work with varieties and schemes defined over the complex numbers. Most of our results should remain true over an algebraically closed field of arbitrary characteristic. When we use the word scheme, we do not necessarily assume that it is of finite type.

2.1 Arcs and jets

Given a variety X, we can consider the following functors from the category of $\mathbf{C}$-algebras to the category of sets:

$$F_X^{\infty}(A) = \operatorname{Hom}\left(\operatorname{Spec} A[\![t]\!],\ X\right), \qquad F_X^{n}(A) = \operatorname{Hom}\left(\operatorname{Spec}\ A[t]/t^{n+1},\ X\right).$$

Both of these functors are representable by schemes, which we denote by X_∞ and X_n respectively. X_∞ is known as the *arc space* of X and X_n as the *n-th jet scheme* of X. The natural projections $\psi_n : X_\infty \to X_n$ are known as *truncation maps*.

The assignment $X \mapsto X_\infty$ is functorial: each morphism $f : X' \to X$ induces by composition a morphism $f_\infty : X'_\infty \to X_\infty$, and $(g \circ f)_\infty = f_\infty \circ g_\infty$. As a consequence, if G is a group scheme, so is G_∞, and if X has an action by G, the arc space X_∞ has an action by G_∞. Analogous statements hold for the jet schemes.

2.2 Contact loci and valuations

A constructible subset $\mathcal{C} \subset X_\infty$ is called *thin* if one can find a proper subscheme $Y \subset X$ such that $\mathcal{C} \subset Y_\infty$. Constructible subsets which are not thin are called *fat*. A *cylinder* in X_∞ is a set of the form $\psi_n^{-1}(C)$, for some constructible set $C \subset X_n$. On a smooth variety, cylinders are fat, but in general a cylinder might be contained in S_∞, where $S = \mathrm{Sing}(X) \subset X$ is the singular locus.

An arc $\alpha \in X_\infty$ induces a morphism $\alpha : \operatorname{Spec} K[\![t]\!] \to X$, where K is the residue field of α. Given an ideal $\mathcal{I} \subset \mathcal{O}_X$, its pull-back $\alpha^*(\mathcal{I}) \subset K[\![t]\!]$ is of the form (t^e) for some non-negative integer e. We call this integer the *order* of α along $\mathcal{I}$ and denote it by $\mathrm{ord}_\alpha(\mathcal{I})$. Given a collection of ideals $I = (\mathcal{I}_1, \dots, \mathcal{I}_r)$ and a multi-index $\mu = (m_1, \dots, m_r) \in \mathbf{Z}_{\geq 0}$ we introduce the *contact locus*:

$$\mathrm{Cont}^{=\mu}(I) = \{ \ \alpha \in X_\infty \ \mid \ \mathrm{ord}_\alpha(\mathcal{I}_j) = m_j \text{ for all } j \ \},$$

$$\mathrm{Cont}^{\mu}(I) = \{ \ \alpha \in X_\infty \ \mid \ \mathrm{ord}_\alpha(\mathcal{I}_j) \geq m_j \text{ for all } j \ \}.$$

Notice that contact loci are cylinders.

Let $\mathcal{C} \subset X_\infty$ be an irreducible fat set. Then $\mathcal{C}$ contains a generic point $\gamma \in \mathcal{C}$ which we interpret as a morphism $\gamma : \operatorname{Spec} K[\![t]\!] \to X$, where K is the residue field of γ. Let η be the generic point of $\operatorname{Spec} K[\![t]\!]$. Since $\mathcal{C}$ is fat, $\gamma(\eta)$ is the generic point of X, and we get an inclusion of fields

$$\mathbf{C}(X) \to K(\!(t)\!).$$

The composition of this inclusion with the canonical valuation on $K(\!(t)\!)$ is a valuation on $\mathbf{C}(X)$, which we denote by $\nu_{\mathcal{C}}$.

A valuation ν of $\mathbf{C}(X)$ is called *divisorial* if it is of the form $q \cdot \mathrm{val}_E$, where q is a positive integer and E is a prime divisor on a variety X' birational to X. An irreducible fat set $\mathcal{C} \subset X_\infty$ is said to be *divisorial* if the corresponding valuation $\nu_{\mathcal{C}}$ is divisorial. In (Ishii, 2008) it is shown that the union of all divisorial sets corresponding to a given valuation ν is itself a divisorial set defining ν (in fact it is an irreducible component of a contact locus). These unions are called *maximal divisorial sets*. There is a one to one correspondence between divisorial valuations and maximal divisorial sets.

2.3 Discrepancies

Let X be a variety of dimension n. The *Nash blowing-up* of X, denoted $\widehat{X}$, is defined as the closure of X_{reg} in $\mathbf{P}_X(\Omega_X^n)$; it is equipped with a tautological line bundle $\mathcal{O}_{\mathbf{P}_X(\Omega_X^n)}(1)|_{\widehat{X}}$, which we denote by $\widehat{K}_X$ and call the *Mather canonical line bundle* of X. When X is smooth, $X = \widehat{X}$ and $K_X = \widehat{K}_X$.

When Y is a smooth variety and $f : Y \to X$ is a birational morphism that factors through the Nash blowing-up, we define the *relative Mather canonical divisor* of f as the unique effective divisor supported on the exceptional locus of f and linearly equivalent to $K_Y - \widehat{K}_X$; we denote it by $\widehat{K}_{Y/X}$.

Let ν be a divisorial valuation of X. Then we can find a smooth variety Y and a birational map $Y \to X$ factoring through the Nash blowing-up of X, such that $\nu = q \cdot \mathrm{val}_E$ for some prime divisor $E \subset Y$. We define the *Mather discrepancy* of X along ν as

$$\widehat{k}_\nu(X) = q \cdot \mathrm{ord}_E \left(\widehat{K}_{Y/X} \right).$$

This definition is independent of the choice of resolution Y.

Following (de Fernex et al., 2008), Mather discrepancies can be computed using the arc space. More precisely, given a divisorial valuation ν with multiplicity q, let $\mathcal{C}_\nu \subset X_\infty$ be the corresponding maximal divisorial set. Then

$$\mathrm{codim}(\mathcal{C}_\nu, X_\infty) = \widehat{k}_\nu(X) + q.$$

2.4 Motivic integration

Let $\mathcal{M}_0$ be the Grothendieck ring of algebraic varieties over $\mathbf{C}$. In (Denef and Loeser, 1999), the authors introduce a certain completion of a localization of $\mathcal{M}_0$, which we denote by $\mathcal{M}$. Also, for each variety X over $\mathbf{C}$, they define a measure μ_X on X_∞ with values in

$\mathcal{M}$. This measure is known as the *motivic measure* of X. The following properties hold for $\mathcal{M}$ and the measures μ_X:

1. There is a canonical ring homomorphism $\mathcal{M}_0 \to \mathcal{M}$. In particular, for each variety X one can associate an element $[X] \in \mathcal{M}$, and the map $X \mapsto [X]$ is additive (meaning that $[X] = [Y] + [U]$, where $Y \subset X$ is a closed subvariety and $U = X \setminus Y$).

2. The element $[\mathbf{A}^1] \in \mathcal{M}$ has a multiplicative inverse. We write $\mathbf{L} = [\mathbf{A}^1]$.

3. Both the Euler characteristic and the Hodge-Deligne polynomial, considered as ring homomorphisms with domain $\mathcal{M}_0$, extend to homomorphisms

$$\chi : \mathcal{M} \to \mathbf{R}, \qquad E : \mathcal{M} \to \mathbf{Z}((u,v)),$$

   where $\chi(\mathbf{L}) = 1$ and $E(\mathbf{L}) = uv$.

4. Constructible sets in X_∞ are μ_X-measurable. In particular, thin sets, fat sets, cylinders, and contact loci are all measurable.

5. If X is smooth, $\mu_X(X_\infty) = [X]$.

6. A thin set has measure zero.

7. Let $\mathcal{C} \subset X_\infty$ be a cylinder in X_∞. Then the truncations $\psi_n(\mathcal{C}) \subset X_n$ are of finite type, so they define elements $[\psi_n(\mathcal{C})] \in \mathcal{M}$. Then

$$\mu_X(\mathcal{C}) = \lim_{n\to\infty} [\psi_n(\mathcal{C})] \cdot \mathbf{L}^{-nd}$$

where d is the dimension of X. Furthermore, if $\mathcal{C}$ does not intersect $(X_{\text{sing}})_\infty$, then $[\psi_n(\mathcal{C})] \cdot \mathbf{L}^{-nd}$ stabilizes for n large enough.

8. Given an ideal $\mathcal{I} \subset \mathcal{O}_X$, we define a function $|\mathcal{I}|$ on X_∞ with values on $\mathcal{M}$ via

$$|\mathcal{I}|(\alpha) = \mathbf{L}^{-\operatorname{ord}_\alpha(\mathcal{I})} \qquad \alpha \in X_\infty.$$

Notice that $\operatorname{ord}_\beta(\mathcal{I}) = \infty$ if and only if $\beta \in \operatorname{Zeroes}(\mathcal{I})_\infty$, so $|\mathcal{I}|$ is only defined up to a measure zero set. Then $|\mathcal{I}|$ is μ_X-integrable and

$$\int_{X_\infty} |\mathcal{I}| \, d\mu_X = \sum_{p=0}^{\infty} [\operatorname{Cont}^{=p}(\mathcal{I})] \cdot \mathbf{L}^{-p}.$$

9. Let $f : Y \to X$ be a birational map factoring through the Nash blowing-up of X, and assume Y smooth. Let $\operatorname{Jac}(f)$ be the ideal of the relative Mather canonical divisor $\widehat{K}_{Y/X}$. Then $(f_\infty)^*(\mu_X) = |\operatorname{Jac}(f)| \cdot \mu_Y$; in other words, for a measurable set $\mathcal{C} \subset X_\infty$, and a μ_X-integrable function φ,

$$\int_{\mathcal{C}} \varphi \; d\mu_X = \int_{f_\infty^{-1}(\mathcal{C})} (\varphi \circ f_\infty) \;\; |\operatorname{Jac}(f)| \;\; d\mu_Y.$$

This is known as the *change of variables formula* for motivic integration.

CHAPTER 3

ARCS ON GENERIC DETERMINANTAL VARIETIES

3.1 Generic determinantal varieties

In this section we review briefly the definition and basic properties of generic determinantal varieties, the main objects of study of this chapter. These results are well known, as determinantal varieties have been studied extensively in the literature. A very comprehensive development of this theory is carried out in (Bruns and Vetter, 1988). Detailed proofs and explanations of the statements in this section can be found there.

Definition 3.1.1 (Generic determinantal varieties)**.** Let $M = \mathbf{A}^{r\cdot s}$ be the space of matrices with r rows and s columns, and assume that $r \leq s$. The *generic determinantal variety* of rank k, which we denote by D^k, is the subvariety of M containing all matrices with rank at most k.

Let $\mathcal{O}_M$ be the ring of regular functions on M; it is a polynomial ring, whose generators correspond to the entries of a generic $r \times s$ matrix.

$$\mathcal{O}_M = \mathbf{C}[x_{00}, \ldots, x_{rs}], \qquad \begin{pmatrix} x_{00} & x_{01} & \cdots & x_{0s} \\ x_{10} & x_{11} & \cdots & x_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ x_{r0} & x_{r1} & \cdots & x_{rs} \end{pmatrix}.$$

The different minors of this generic matrix give functions in $\mathcal{O}_M$, and we can consider the ideals $\mathcal{I}_{D^k} \subset \mathcal{O}_M$ generated by the $(k+1) \times (k+1)$ minors. One can show that $\mathcal{I}_{D^k}$ is prime (Bruns and Vetter, 1988, Thm. 2.10), and that its zero set is D^k, the collection of matrices with rank at most k (Bruns and Vetter, 1988, §1.C). In particular D^k is an irreducible affine algebraic variety.

Since a minor of size $(k+1) \times (k+1)$ can be expressed in terms of minors of size $k \times k$, we have a chain of ideals

$$\mathcal{O}_M \supset \mathcal{I}_{D^0} \supset \mathcal{I}_{D^1} \supset \cdots \supset \mathcal{I}_{D^r} = 0,$$

corresponding to a filtration

$$D^0 \subset D^1 \subset \cdots \subset D^r = M.$$

Moreover, it can be shown that $\mathcal{I}_{D^{k-1}}$ is the Jacobian ideal of $\mathcal{I}_{D^k}$ (Bruns and Vetter, 1988, Prop. 1.1, §6.B). In other words, D^k is singular, and its singular locus is exactly D^{k-1}.

Consider the algebraic group $G = \mathrm{GL}_r \times \mathrm{GL}_s$. It has a natural algebraic action on M via change of basis:

$$(g,h) \cdot A \;=\; g\,A\,h^{-1}, \qquad\qquad g \in \mathrm{GL}_r, \quad h \in \mathrm{GL}_s, \quad A \in M.$$

The orbits of this action are the sets of the form $D^k \setminus D^{k-1}$. Moreover, the only invariant prime ideals in $\mathcal{O}_M$ are $\mathcal{I}_{D^0}, \mathcal{I}_{D^1}, \dots, \mathcal{I}_{D^r}$.

3.2 Orbit decomposition of the arc space

As we saw in the previous section, the group $G = \mathrm{GL}_r \times \mathrm{GL}_s$ acts on the space M of $r \times s$ matrices. In fact the group G acts on all the determinantal varieties D^k, as these are the orbit closures of the action of G on M.

The group G is a reductive algebraic group. In particular it is an algebraic variety, and we can consider its arc space G_∞ and its jet schemes G_n. Moreover, since the assignment that sends a scheme to its associated arc space (or jet schemes) is functorial, both G_∞ and G_n are group schemes, and we have induced actions on the arc spaces and jet schemes of M and D^k:

$$G_\infty \times M_\infty \to M_\infty, \qquad G_\infty \times D^k_\infty \to D^k_\infty,$$

$$G_n \times M_n \to M_n, \qquad G_n \times D^k_n \to D^k_n.$$

In this section we classify the orbits associated to all of these actions. The notations introduced here will be used through the rest of the chapter.

We start with the action of G_∞ on M_∞. As a set, the arc space M_∞ contains matrices of size $r \times s$ with entries in the power series ring $\mathbf{C}[\![t]\!]$. Analogously, the group $G_\infty = (\mathrm{GL}_r)_\infty \times (\mathrm{GL}_s)_\infty$ is formed by pairs of square matrices with entries in $\mathbf{C}[\![t]\!]$ which are invertible, that is, their determinant is a unit in $\mathbf{C}[\![t]\!]$. Orbits in M_∞ correspond to similarity classes of matrices over the ring $\mathbf{C}[\![t]\!]$, and we can easily classify these using the fact that $\mathbf{C}[\![t]\!]$ is a principal ideal domain.

In order to make the classification more convenient, we will use the language of partitions.

Definition 3.2.1 (Partitions)**.** Given a positive integer p, a *partition* of p is a weakly decreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$ whose sum is p. The number ℓ of integers in the sequence is called the *length* of the partition, and the first integer λ_1 is known as the *highest term*. We allow ourselves to consider the *empty partition*, the only partition of length 0. The set of partitions with length at most r is denoted by Λ_r, and the set of partitions with length at most r and highest term at most n is denoted by $\Lambda_{r,n}$.

Definition 3.2.2 (Extended partitions)**.** An *extended partition* is a partition where we allow some of the integers to be infinity. More formally, we endow the set $\overline{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$ with the natural order where ∞ is the maximum, and we say that an extended partition is a weakly decreasing sequence of elements in $\overline{\mathbf{N}}$. An extended partition which is not a partition is said to be a partition of infinity. The set of extended partitions of length at most r is denoted by $\overline{\Lambda}_r$.

Remark 3.2.3. It will be convenient to consider partitions containing trailing zeros. For this reason, given an extended partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and an integer $k > \ell$, we set $\lambda_k = 0$.

For the following definition, recall that M is the space of matrices with r rows and s columns, and that we assume $r \leq s$.

Definition 3.2.4 (Orbit associated to a partition)**.** Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \overline{\Lambda}_r$ be an extended partition with length at most r. Recall that we set $\lambda_k = 0$ for $k > \ell$. Consider the following diagonal matrix in M_∞:

$$\delta_\lambda = \begin{pmatrix} 0 & \cdots & 0 & t^{\lambda_1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & t^{\lambda_2} & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & t^{\lambda_r} \end{pmatrix}$$

(we use the convention that $t^\infty = 0$). Then the G_∞-orbit of the matrix δ_λ is called the orbit in M_∞ *associated* to the partition λ, and it is denoted by $\mathcal{C}_\lambda$.

Proposition 3.2.5 (Orbits in M_∞)**.** *Every G_∞-orbit of M_∞ is of the form $\mathcal{C}_\lambda$ for some extended partition $\lambda \in \overline{\Lambda}_r$. An orbit $\mathcal{C}_\lambda$ is contained in D^k_∞ if and only if the associated extended partition λ contains at least $r-k$ leading infinities, i.e. $\lambda_1 = \cdots = \lambda_{r-k} = \infty$. In particular, $M_\infty \setminus D^{r-1}_\infty$ is the union of the orbits corresponding to regular partitions, and the orbits in $D^k_\infty \setminus D^{k-1}_\infty$ are in bijection with Λ_k. Moreover, the orbit corresponding to the empty partition is the arc space $\left(M \setminus D^{r-1}\right)_\infty$.*

Proof. As mentioned above, M_∞ is the set of $r \times s$-matrices with coefficients in the ring $\mathbf{C}[\![t]\!]$, and the group G_∞ acts on M_∞ via row and column operations, also with coefficients in $\mathbf{C}[\![t]\!]$. Using Gaussian elimination and the fact that $\mathbf{C}[\![t]\!]$ is a principal ideal domain, we see that each G_∞-orbit in M_∞ contains a diagonal matrix, where the diagonal entries are powers of t or zeroes. Think of the diagonal zeroes as powers t^∞. After permuting columns and rows, we can assume that the exponents of these powers form a weakly decreasing sequence when read from the upper-left corner to the lower-right corner. Moreover, the usual structure theorems for finitely generated modules over principal ideal domains guarantee that each orbit contains a unique diagonal matrix in this form. This shows that the set of G_∞-orbits in M_∞ is in bijection with $\overline{\Lambda}_r$.

The ideal defining D^k in M is generated by the minors of size $(k+1) \times (k+1)$. Let $\lambda \in \overline{\Lambda}_r$ be an extended partition of length at most r and consider δ_λ as in Definition 3.2.4. The $(k+1) \times (k+1)$ minors of δ_λ are either zero or of the form $\prod_{i \in I} t^{\lambda_i}$, where I is a subset of $\{1, \ldots, r\}$ with $k+1$ elements. For all of the minors to be zero, we need at least $r-k$ infinities in the set $\{\lambda_1, \ldots, \lambda_r\}$ (recall that $t^\infty = 0$). In other words, δ_λ is contained in D^k_∞ if and only if λ contains $r-k$ leading infinities.

The variety D^k is invariant under the action of G, so D^k_∞ is invariant under the action of G_∞. In particular the orbit $\mathcal{C}_\lambda$ is contained in D^k_∞ if and only if δ_λ is. The rest of the proposition follows immediately. □

Proposition 3.2.6 (Orbits and contact loci)**.** *The contact locus* $\mathrm{Cont}^p(D^k)$ *is invariant under the action of* G_∞*, and the orbits contained in* $\mathrm{Cont}^p(D^k)$ *correspond to those extended partitions* $\lambda \in \overline{\Lambda}_r$ *whose last* $k+1$ *terms add up to at least* p*:*

$$\lambda_{r-k} + \cdots + \lambda_r \geq p.$$

Proof. The truncations maps from the arc space to the jet schemes are in fact natural transformations of functors. This means that we have the following natural diagram:

$$\begin{tikzcd} G_\infty \times M_\infty \arrow[r] \arrow[d] & M_\infty \arrow[d] \\ G_n \times M_n \arrow[r] & M_n \end{tikzcd}$$

Since D^k is G-invariant, D^k_n is G_n-invariant, so $\mathrm{Cont}^p(D^k)$ (the inverse image of D^k_{p-1} under the truncation map) is G_∞-invariant. In particular, an orbit $\mathcal{C}_\lambda$ is contained in $\mathrm{Cont}^p(D^k)$ if and only if its base point δ_λ is. The order of vanishing of $\mathcal{I}_{D^k}$ along δ_λ is $\lambda_{r-k} + \cdots + \lambda_r$ (recall that $\mathcal{I}_{D^k}$ is generated by the minors of size $(k+1) \times (k+1)$ and that $\lambda_1 \geq \cdots \geq \lambda_r$). Hence δ_λ belongs to $\mathrm{Cont}^p(D^k)$ if and only if $\lambda_{r-k} + \cdots + \lambda_r \geq p$, and the proposition follows. $\square$

Proposition 3.2.7 (Orbits are cylinders)**.** *Let $\lambda \in \overline{\Lambda}_r$ be an extended partition, and let $\mathcal{C}_\lambda$ be the associated orbit in M_∞. If λ is a partition, $\mathcal{C}_\lambda$ is a cylinder of M_∞. More generally, let $r-k$ be the number of infinite terms of λ. Then $\mathcal{C}_\lambda$ is a cylinder of D^k_∞.*

Proof. Assume that λ is an extended partition with $r-k$ leading infinities, and consider the following cylinders in M_∞:

$$A_\lambda = \mathrm{Cont}^{\lambda_r}(D^0) \cap \mathrm{Cont}^{\lambda_r+\lambda_{r-1}}(D^1) \cap \mathrm{Cont}^{\lambda_r+\cdots+\lambda_{r-k}}(D^k),$$

$$B_\lambda = \mathrm{Cont}^{\lambda_r+1}(D^0) \cup \mathrm{Cont}^{\lambda_r+\lambda_{r-1}+1}(D^1) \cup \mathrm{Cont}^{\lambda_r+\cdots+\lambda_{r-k}+1}(D^k).$$

By Propositions 3.2.5 and 3.2.6, we know that

$$\mathcal{C}_\lambda = (A_\lambda \setminus B_\lambda) \cap D^k_\infty.$$

Hence $\mathcal{C}_\lambda$ is a cylinder in D^k_∞, as required. $\square$

We now study the jet schemes G_n and M_n. As in the case of the arc space, elements in G_n and M_n are given by matrices, but now the coefficients lie in the ring $\mathbf{C}[t]/(t^{n+1})$.

Definition 3.2.8. Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \Lambda_{r,n+1}$ be an extended partition with length at most r and highest term at most $n+1$. Then the diagonal matrix δ_λ considered in Definition 3.2.4 gives an element of the jet scheme M_n. The G_n-orbit of δ_λ is called the orbit of M_n *associated* to λ and it is denoted by $\mathcal{C}_{\lambda,n}$.

Proposition 3.2.9 (Orbits in M_n)**.** *Every G_n-orbit of M_n is of the form $\mathcal{C}_{\lambda,n}$ for some partition $\lambda \in \Lambda_{r,n+1}$. An orbit $\mathcal{C}_{\lambda,n}$ is contained in D^k_n if and only if the associated partition contains at least $r-k$ terms equal to $n+1$. In particular, the set of orbits in $D^k_n \setminus D^{k-1}_n$ is in bijection with $\Lambda_{k,n}$.*

Proof. This can be proven in the same way as Proposition 3.2.5. The only difference is that we now work with a principal ideal ring $\mathbf{C}[t]/(t^{n+1})$, as opposed to with the principal ideal domain $\mathbf{C}[\![t]\!]$, but the domain condition played no role in the proof of Proposition 3.2.5. Alternatively, one can notice that $\mathbf{C}[t]/(t^{n+1})$ is a quotient of $\mathbf{C}[\![t]\!]$, so modules over $\mathbf{C}[t]/(t^{n+1})$ correspond to modules over $\mathbf{C}[\![t]\!]$ with the appropriate annihilator, and one can reduce the problem of classifying G_n-orbits in M_n to classifying G_∞-orbits in M_∞ with bounded exponents. $\square$

Definition 3.2.10. Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be an extended partition, and let n be a nonnegative integer. Then the *truncation* of λ to level n is the partition $\overline{\lambda} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_\ell)$ where

$$\overline{\lambda}_i = \min(\lambda_i, n).$$

Proposition 3.2.11 (Truncation of orbits)**.** *Let $\lambda \in \overline{\Lambda}_r$ be an extended partition, and let $\overline{\lambda}$ be its truncation to level $n+1$. Then the image of $\mathcal{C}_\lambda$ under the natural truncation map $M_\infty \to M_n$ is $\mathcal{C}_{\overline{\lambda},n}$. Conversely, fix a partition $\overline{\lambda} \in \Lambda_{r,n+1}$, and let $\Gamma \subset \overline{\Lambda}_r$ be the set of extended partitions whose truncation to level $n+1$ is $\overline{\lambda}$. Then the inverse image of $\mathcal{C}_{\overline{\lambda},n}$ under the truncation map is the union of the orbits of M_∞ corresponding to the extended partitions in Γ.*

Proof. Notice that $\delta_{\overline{\lambda}} \in M_n$ is the truncation of $\delta_\lambda \in M_\infty$. Then the fact that the truncation of $\mathcal{C}_\lambda = G_\infty \cdot \delta_\lambda$ equals $\mathcal{C}_{\overline{\lambda},n} = G_n \cdot \delta_{\overline{\lambda}}$ is an immediate consequence of the fact that the truncation map is a natural transformation of functors (see the proof of Proposition 3.2.6). Conversely, if λ and λ' have different truncations, the G_n-orbits $\mathcal{C}_{\overline{\lambda},n}$ and $\mathcal{C}_{\overline{\lambda}',n}$ are different, so $\mathcal{C}_{\lambda'}$ is not in the fiber of $\mathcal{C}_{\overline{\lambda},n}$. $\square$

3.3 The orbit poset

After obtaining a classification of the orbits of the action of $G_\infty = (\mathrm{GL}_r)_\infty \times (\mathrm{GL}_s)_\infty$ on M_∞ and D^k_∞, we start the study of their geometry. The first basic question is the following:

how are these orbits positioned with respect to each other inside the arc space M_∞? We can make this precise by introducing the notion of orbit poset.

Definition 3.3.1 (Orbit poset)**.** Let $\mathcal{C}$ and $\mathcal{C}'$ be two G_∞-orbits in M_∞. We say that $\mathcal{C}$ *dominates* $\mathcal{C}'$, and denote it by $\mathcal{C}' \leq \mathcal{C}$, if $\mathcal{C}'$ is contained in the Zariski closure of $\mathcal{C}$. The relation of dominance defines a partial order on the set of G_∞-orbits of M_∞. The pair $(M_\infty/G_\infty, \leq)$ is known as the *orbit poset* of M_∞.

The goal of this section is to understand the structure of the orbit poset.

Our strategy is as follows. As we saw in Section 3.2, the collection of G_∞-orbits in M_∞ is in bijection with $\overline{\Lambda}_r$, the set of extended partitions of length at most r. Motivated by this, we will define a natural order in the set of partitions, giving rise to what we call the *partition poset*. This new poset is a purely combinatorial object, which we can understand completely, and we will use combinatorial techniques and some geometry of the arc space to show that the orbit poset and the partition poset are isomorphic.

There are different ways of defining interesting partial orders in the space of partitions. The most common one is probably *containment of partitions*, which is given by

$$\lambda \subseteq \lambda' \qquad \Longleftrightarrow \qquad \lambda_i \leq \lambda'_i \quad \forall i \geq 1.$$

Another possibility is the *dominance* order, sometimes also called *majorization* order:

$$\lambda \trianglelefteq \lambda' \qquad \Longleftrightarrow \qquad \lambda_1 + \cdots + \lambda_i \leq \lambda'_1 + \cdots + \lambda'_i \quad \forall i \geq 1.$$

In our case we need to consider a slight variation of the dominance order; for convenience we will give this new order a name: we call it the order of *subordination*.

Definition 3.3.2 (Partition poset)**.** Let $\lambda, \lambda' \in \overline{\Lambda}_r$ be two extended partitions of length at most r. We say that λ is *subordinate* to λ', and denote it by $\lambda \preceq \lambda'$, if

$$\lambda_r + \lambda_{r-1} + \cdots + \lambda_{r-i} \quad \leq \quad \lambda'_r + \lambda'_{r-1} + \cdots + \lambda'_{r-i} \qquad \forall i \in \{0, \ldots, r-1\}.$$

Subordination defines a partial order on $\overline{\Lambda}_r$. The pair $(\overline{\Lambda}_r, \preceq)$ is known as the *partition poset* of length r.

Remark 3.3.3. The definition above is independent of the integer r, in the sense that for $r' \geq r$ the order of subordination of $\overline{\Lambda}_r$ is the restriction of the order of subordination of $\overline{\Lambda}_{r'}$ via the natural inclusion $\overline{\Lambda}_r \subset \overline{\Lambda}_{r'}$. This justifies the absence of r in the notation $\lambda \preceq \lambda'$.

Our goal is to prove that the bijection that maps a partition to its associated orbit in M_∞ is in fact an order-reversing isomorphism between the partition poset and the orbit poset. At this stage it is not hard to show that one of the directions of this bijection reverses the order.

Proposition 3.3.4 (Domination implies subordination)**.** *Let $\lambda, \lambda' \in \overline{\Lambda}_r$ be two extended partitions of length at most r, and let $\mathcal{C}_\lambda$ and $\mathcal{C}_{\lambda'}$ be the associated orbits in M_∞. If $\mathcal{C}_\lambda$ dominates $\mathcal{C}_{\lambda'}$, then λ is subordinate to λ'.*

$$\mathcal{C}_\lambda \geq \mathcal{C}_{\lambda'} \qquad \Longrightarrow \qquad \lambda \preceq \lambda'$$

Proof. From Proposition 3.2.6 we get that $\mathcal{C}_\lambda \subset \operatorname{Cont}^p(D^k)$, where $p = \lambda_r + \cdots + \lambda_{r-k}$. Since a contact locus is always Zariski closed, if $\mathcal{C}_\lambda$ dominates $\mathcal{C}_{\lambda'}$, we also know that $\mathcal{C}_{\lambda'} \subset \operatorname{Cont}^p(D^k)$. Again by Proposition 3.2.6, this gives $\lambda'_r + \cdots + \lambda'_{r-k} \geq p$, as required. $\square$

We now proceed to prove the converse to Proposition 3.3.4. Given two extended partitions $\lambda, \lambda' \in \overline{\Lambda}_r$ with $\lambda \preceq \lambda'$, we need to show that the closure of $\mathcal{C}_\lambda$ contains $\mathcal{C}_{\lambda'}$. We will exhibit this containment by producing a "path" in the arc space M_∞ whose general point is in $\mathcal{C}_\lambda$ but specializes to a point in $\mathcal{C}_{\lambda'}$. These types of "paths" are known as *wedges*.

Definition 3.3.5 (Wedge)**.** Let X be a scheme over $\mathbf{C}$. A *wedge* w on X is a morphism of schemes $w : \operatorname{Spec} \mathbf{C}\llbracket s, t \rrbracket \to X$. Given a wedge w, one can consider the diagram

$$\begin{tikzcd}
\operatorname{Spec} \mathbf{C}\llbracket t \rrbracket \arrow[dr, "s \mapsto 0"] \arrow[drr, bend left, "w_0"] & & \\
& \operatorname{Spec} \mathbf{C}\llbracket s, t \rrbracket \arrow[r, "w"] & X. \\
\operatorname{Spec} \mathbf{C}(\!(s)\!)\llbracket t \rrbracket \arrow[ur] \arrow[urr, bend right, "w_s"'] & &
\end{tikzcd}$$

The map w_0 is known as the *special arc* of w, and w_s as the *generic arc* of w.

One would like to produce one wedge for each relation of the type $\lambda \prec \lambda'$. This can be done, but the resulting expression for w is overly complicated, and not very illuminating. Instead, we will produce wedges only for a special class of relations, what we call the *covering relations*, and then show that these generate the partition poset in a suitable sense.

As a bonus of this approach, the analysis of the covering relations gives information about the structure of the poset.

Definition 3.3.6 (Covering relations)**.** Let $\lambda, \lambda' \in \overline{\Lambda}_r$ be two extended partitions. We say that λ' is a *cover* of λ with respect to the order $\preceq$ if $\lambda \prec \lambda'$ and there is no partition in between:

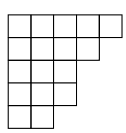
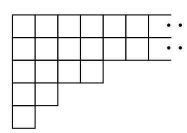
$$\nexists \lambda'' \quad \text{s.t.} \quad \lambda \prec \lambda'' \prec \lambda'.$$

The idea behind this definition is that covers are the simplest relations in a poset, and in a reasonable poset all upper bounds can be expressed as a sequence of covers. Unfortunately, for the poset of extended partitions, covers are not enough, and one also needs to consider what we will call "hyper-covers". We will show this in Proposition 3.3.10, but first we need to classify covers in the partition poset. The following construction is an adaptation of a similar well-known result for the dominance order.

Definition 3.3.7 (Hyper-covers in $\overline{\Lambda}_r$)**.** Let $\lambda, \lambda' \in \overline{\Lambda}_r$ be two extended partitions, and assume that $\lambda \prec \lambda'$. We say that λ' is a *hyper-cover* of λ if one of the following statements is true.

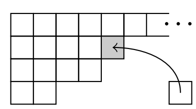
$$\begin{array}{llllll}
\text{Type 1.} & \exists i & \text{s.t.} & \lambda'_k = \lambda_k \quad \forall k \neq i & \text{and} & \lambda'_i = \infty,\ \lambda_i < \infty \\
\text{Type 2.} & \exists i & \text{s.t.} & \lambda'_k = \lambda_k \quad \forall k \neq i & & \\
 & & \text{and} & \lambda_k = \infty \quad \forall k < i & \text{and} & \lambda'_i = \lambda_i + 1 \\
\text{Type 3.} & \exists i & \text{s.t.} & \lambda'_k = \lambda_k \quad \forall k \neq i, i+1 & \text{and} & \lambda'_i = \lambda_i - 1,\ \lambda'_{i+1} = \lambda_{i+1} + 1 \\
\text{Type 4.} & \exists i < j & \text{s.t.} & \lambda'_k = \lambda_k \quad \forall k \neq i, j & & \\
 & & \text{and} & \lambda'_k = \lambda'_{k'} \quad \forall i \leq k, k' \leq j & \text{and} & \lambda'_i = \lambda_i - 1,\ \lambda'_j = \lambda_j + 1
\end{array}$$

Remark 3.3.8. To understand the content of Definition 3.3.7, it is helpful to visualize partitions as Young diagrams. A Young diagram is a graphical representation of a partition; it is a collection of boxes, arranged in left-justified rows, with weakly decreasing row sizes. To each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of size r there is a unique Young diagram with r rows and whose i-th row has size λ_i. In order to consider extended partitions, we also allow Young diagrams to have infinitely many boxes in some rows. For example:

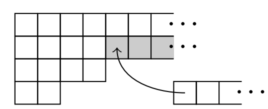
$(5, 4, 3, 3, 2) \quad =$ 
$(\infty, \infty, 4, 2, 1) \quad =$ 

In view of Definition 3.3.7, we can understand how Young diagrams get transformed when we go from a partition λ to one of its hyper-covers λ'. Hyper-covers of types 3 and

4 preserve the number of boxes in the diagrams, whereas the ones of types 1 and 2 add boxes. In the cases of type 1 and type 2, we only add boxes to the top-most row with finite size; for type 2 we add one box; for type 1 we add infinitely many.

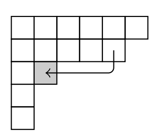


Type 2.

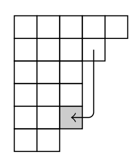


Type 1.

Types 3 and 4 can be interpreted as moves. They take the right-most box from row i and place it in a lower row. For type 3 we move one row down and as many columns left as we want; for type 4 we move one column left and many rows down.



Type 3.



Type 4.

Proposition 3.3.9. *The covers in the partition poset are exactly the hyper-covers of types 2, 3, and 4.*

We will prove Proposition 3.3.9 in several steps. First we will show that each of the three types above actually gives a cover. Then we will give an algorithm that for each upper bound $\lambda \prec \lambda'$ produces a sequence $\lambda \preceq \mu \prec \lambda'$ such that $\mu \prec \lambda'$ is hyper-cover of type 2, 3, or 4.

Proof of 3.3.9. Step 1: type 2 is a cover. Let $\lambda \prec \lambda'$ be a hyper-cover of type 2, and let i be such that $\lambda'_i = \lambda_i + 1$. Assume that $\lambda \preceq \mu \prec \lambda'$ for some partition μ. By the definition of type 2 hyper-cover, it is clear that $\lambda_k = \lambda'_k = \mu_k$ for $k \neq i$. Moreover, since $\lambda \preceq \mu \prec \lambda'$, we get that $\lambda_i \leq \mu_i < \lambda'_i$. Since $\lambda'_i = \lambda_i + 1$ we must have, $\lambda_i = \mu_i$. Hence $\lambda = \mu$, as required. □

Proof of 3.3.9. Step 2: type 3 is a cover. Let $\lambda \prec \lambda'$ be a hyper-cover of type 3, and let i be such that $\lambda'_i = \lambda_i - 1$ and $\lambda'_{i+1} = \lambda_{i+1} + 1$. Assume that $\lambda \preceq \mu \prec \lambda'$ for some partition μ. By the definition of type 3 hyper-cover, it is clear that $\lambda_k = \lambda'_k = \mu_k$ for $k \neq i, i+1$. Moreover, since $\lambda \preceq \mu \prec \lambda'$, we get that $\lambda_{i+1} \leq \mu_{i+1} < \lambda_{i+1} + 1$ and $\mu_i + \mu_{i+1} = \lambda_i + \lambda_{i+1}$. Hence $\lambda = \mu$, as required. □

Proof of 3.3.9. Step 3: type 4 is a cover. Let $\lambda \prec \lambda'$ be a hyper-cover of type 4, and let $i < j$ be such that $\lambda'_i = \lambda_i - 1$ and $\lambda'_j = \lambda_j + 1$. Assume that $\lambda \preceq \mu \preceq \lambda'$ for some partition μ. By the definition of type 4 hyper-cover, it is clear that $\lambda_k = \lambda'_k = \mu_k$ for $k \notin \{i, i+1, \ldots, j\}$. Moreover, since $\lambda \preceq \mu \preceq \lambda'$, we get that

$$\begin{array}{ccccc}
\lambda'_j - 1 & \leq & \mu_j & \leq & \lambda'_j, \\
\lambda'_j + \lambda'_{j-1} - 1 & \leq & \mu_j + \mu_{j-1} & \leq & \lambda'_j + \lambda'_{j-1}, \\
 & & \vdots & & \\
\lambda'_j + \cdots + \lambda'_{i+1} - 1 & \leq & \mu_j + \cdots + \mu_{i+1} & \leq & \lambda'_j + \cdots + \lambda'_{i+1}, \\
\lambda'_j + \cdots + \lambda'_i & = & \mu_j + \cdots + \mu_i & = & \lambda'_j + \cdots + \lambda'_i.
\end{array}$$

Another condition in the definition of type 4 hyper-cover says that $\lambda'_i = \lambda'_{i+1} = \cdots = \lambda'_j$. Writing $\mu_k = \lambda'_j - \mu'_k$ we obtain

$$\begin{array}{rcccl} -1 & \leq & -\mu'_j & \leq & 0, \\ -1 & \leq & -\mu'_j - \mu'_{j-1} & \leq & 0, \\ & & \vdots & & \\ -1 & \leq & -\mu'_j - \cdots - \mu'_{i+1} & \leq & 0, \\ 0 & = & -\mu'_j - \cdots - \mu'_i & = & 0, \end{array}$$

and

$$\mu'_i \leq \mu'_{i+1} \leq \cdots \leq \mu'_j.$$

The only possibilities for $(\mu'_i, \ldots, \mu'_j)$ are $(1, 0, \ldots, 0, -1)$ and $(0, \ldots, 0)$, which imply $\mu = \lambda'$ or $\mu = \lambda$, as required. □

Proof of 3.3.9. Final step: there are no more covers. Let $\lambda, \lambda' \in \overline{\Lambda}_r$ be such that $\lambda \prec \lambda'$. Our goal is to find a hyper-cover μ of λ of type 2, 3 or 4 such that $\lambda \prec \mu \preceq \lambda'$.

If λ' has more infinites than λ, we can take μ to be a hyper-cover of type 2. Therefore we can assume that both λ and λ' are partitions. In this case, if $\lambda_1 + \cdots + \lambda_r < \lambda'_1 + \cdots + \lambda'_r$, we can again take μ to be a hyper-cover of type 2. This way we reduce to the situation where λ and λ' are partitions of the same number p.

Consider:

$$e_k = \lambda_k + \cdots + \lambda_r, \quad k \in \{1, \ldots, r\}$$

$$f_k = \lambda'_k + \cdots + \lambda'_r, \quad k \in \{1, \ldots, r\}$$

$$N = \max\{\ f_k - e_k \ \mid\ k = 1, \ldots, r\ \} > 0$$

$$b = \max\{\ k \ \mid\ N = f_k - e_k\ \} \geq 2$$

$$a = \min\{\ k < b \ \mid\ N = f_j - e_j \ \ \forall j \in [k+1, b]\ \} \geq 1$$

Notice that from the definition of N, a and b we have:

$$\lambda'_k + \cdots + \lambda'_r = N + \lambda_k + \cdots + \lambda_r, \qquad \text{for} \quad a < k \leq b.$$

In particular, if μ is a hyper-cover of λ of type 3 with $a \leq i < b$, or of type 4 with $a \leq i < j \leq b$, we have that $\mu \preceq \lambda'$. We only need to show that it is possible to pick μ with this restrictions.

By construction we have:

$$\lambda_a > \lambda'_a, \qquad \lambda_b < \lambda'_b,$$

$$\lambda_a > \lambda_k = \lambda'_k > \lambda_b \quad \forall k \in (a, b).$$

If $\lambda_{a+1} < \lambda_a - 1$ we can pick μ to be a hyper-cover of type 3 with $i = a$. If $\lambda_b < \lambda_{b-1} - 1$ we can pick μ to be a hyper-cover of type 3 with $i = b - 1$. Otherwise we can pick μ to be a hyper-cover of type 4 with $i = a$ and $j = b$. $\square$

Proposition 3.3.10. *Let $\lambda, \lambda' \in \overline{\Lambda}_r$ be two extended partitions, and assume that $\lambda \prec \lambda'$. Then there exists an increasing sequence*

$$\lambda = \mu^0 \prec \mu^1 \prec \cdots \prec \mu^k = \lambda'$$

of extended partitions such that μ^i is a hyper-cover of μ^{i-1} for all $i \in \{1, \ldots, k\}$. Moreover, if λ and λ' have the same number of infinite terms, the hyper-covers are actually covers.

Proof. If λ' has more infinites that λ, we can find a sequence of hyper-covers of type 1 $\lambda = \mu^0 \prec \mu^1 \prec \cdots \prec \mu^a$ such that $\mu^a \preceq \lambda'$ and μ^a has the same number of infinites as λ'. This reduces the proof to the case where λ and λ' are partitions. In this situation, we can always find a cover $\lambda \prec \mu$ such that $\mu \preceq \lambda'$. Since there are only finitely many partitions in between λ and λ', we can iterate this process and get the result. $\square$

Theorem 3.3.11 (Orbit poset = Partition poset)**.** *The map that sends an extended partition $\lambda \in \overline{\Lambda}_r$ of length at most r to the associated orbit $\mathcal{C}_\lambda$ in M_∞ is an order-reversing isomorphism between the partition poset and the orbit poset.*

$$(M_\infty / G_\infty, \leq)^{\mathrm{op}} \quad \simeq \quad (\overline{\Lambda}_r, \preceq)$$

$$\mathcal{C}_\lambda \geq \mathcal{C}_{\lambda'} \quad \Longleftrightarrow \quad \lambda \preceq \lambda' \quad \Longleftrightarrow \quad \lambda_r + \cdots + \lambda_{r-i} \leq \lambda'_r + \cdots + \lambda'_{r-i} \quad \forall i$$

Proof. By Proposition 3.3.4, we only need to prove that $\mathcal{C}_\lambda \geq \mathcal{C}_{\lambda'}$ when $\lambda \prec \lambda'$. Moreover, using Proposition 3.3.10 we can assume that λ' is a hyper-cover of λ.

Assume first that λ' is a hyper-cover of λ of type 1 or of type 2 and let i be the index such that $\lambda_i < \lambda'_i$. Consider the following wedge on M:

$$w = \begin{pmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & st^{\lambda_i} + t^{\lambda'_i} & & & \\ & & & & t^{\lambda_{i+1}} & & \\ & & & & & \ddots & \\ & & & & & & t^{\lambda_r} \end{pmatrix}.$$

Then the general arc w_s of w is contained in $\mathcal{C}_\lambda$, while the special arc $w_0 = \delta_{\lambda'}$ is contained in $\mathcal{C}'_\lambda$ (see Definition 3.3.5 for the notions of general arc and special arc of a wedge). This means that $\mathcal{C}_{\lambda'}$ intersects the closure of $\mathcal{C}_\lambda$, of in fact $\mathcal{C}_{\lambda'}$ is dominated by $\mathcal{C}_\lambda$.

Assume now that λ' is a hyper-cover of λ of type 3 or of type 4, and let $i < j$ be the indices such that $\lambda'_i = \lambda_i - 1$, $\lambda'_j = \lambda_j + 1$ and $\lambda'_k = \lambda_k$ for $k \neq i, j$. Consider the following wedge:

$$w = \begin{pmatrix} t^{\lambda_1} & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & t^{\lambda_{i-1}} & & & & & & & & \\ & & & \alpha & 0 & \cdots & 0 & \beta & & & \\ & & & 0 & t^{\lambda_{i+1}} & \cdots & 0 & 0 & & & \\ & & & \vdots & \vdots & \ddots & \vdots & \vdots & & & \\ & & & 0 & 0 & \cdots & t^{\lambda_{j-1}} & 0 & & & \\ & & & \gamma & 0 & \cdots & 0 & \delta & & & \\ & & & & & & & & t^{\lambda_{j+1}} & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & t^{\lambda_r} \end{pmatrix}$$

where

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} st^{\lambda_i} + t^{\lambda_i - 1} & t^{\lambda_i - 1} \\ st^{\lambda_j} & st^{\lambda_j} + t^{\lambda_j + 1} \end{pmatrix}.$$

Notice that

$$\operatorname{ord}_t \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \lambda_j, \qquad \operatorname{ord}_t \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \Bigg|_{s=0} = \lambda_j + 1,$$

$$\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = t^{\lambda_i + \lambda_j} \left(1 + st + s^2\right).$$

We see that for the wedge w, the general arc w_s is contained in $\mathcal{C}_\lambda$, while the special arc w_0 is contained in $\mathcal{C}_{\lambda'}$. Therefore $\mathcal{C}_\lambda$ dominates $\mathcal{C}_{\lambda'}$, and the theorem is proven. □

3.4 Irreducible components of jet schemes

In this section we compute the number of irreducible components of the jet schemes of determinantal varieties.

Notation 3.4.1. As it is customary in the theory of partitions, we write $\lambda = (d_1^{a_1} \ldots d_j^{a_j})$ to denote the partition that has a_i copies of the integer d_i. For example $(5,3,3,3,2,1,1) = (5^1\, 3^3\, 2^1\, 1^2)$.

Proposition 3.4.2. *Recall that $D^k \subset M$ denotes the determinantal variety of matrices of size $r \times s$ and rank at most k. Assume that $0 < k < r - 1$, and let $\mathcal{C}$ be an irreducible component* *of* $\mathrm{Cont}^p(D^k) \subset M_\infty$*. Then* $\mathcal{C}$ *contains a unique dense* G_∞*-orbit* $\mathcal{C}_\lambda$*. Moreover,* λ *is a partition (contains no infinite terms) and* $\lambda = (d^{a+r-k}\, e^1)$ *where*

$$p = (a+1)\, d + e, \qquad\qquad 0 \le e < d,$$

and either $e = 0$ *and* $0 \le a \le k$ *or* $e > 0$ *and* $0 \le a < k$*. Conversely, for any partition as above, its associated orbit is dense in an irreducible component of* $\mathrm{Cont}^p(D^k)$*.*

Example 3.4.3. When $r = 8$, $k = 6$, and $p = 5$, the partitions given by the proposition are

$$(5,5), \quad (4,4,1), \quad (3,3,2), \quad (2,2,2,1), \quad (1,1,1,1,1,1).$$

When $r = 5$, $k = 3$, and $p = 5$, we only get

$$(5,5), \quad (4,4,1), \quad (3,3,2), \quad (2,2,2,1).$$

Proof. By Theorem 3.3.11 and Proposition 3.2.6, computing the irreducible components of $\mathrm{Cont}^p(D^k)$ is equivalent to computing the minimal elements (with respect to the order of subordination) among all extended partitions $\lambda \in \overline{\Lambda}_r$ such that

$$\lambda_r + \lambda_{r-1} + \cdots + \lambda_{r-k} \ge p.$$

Let Σ be the set of such partitions. To find minimal elements in Σ it will be useful to keep in mind the covering relations discussed in Proposition 3.3.7.

First notice that all minimal elements in Σ must be partitions. Indeed, given an element $\lambda \in \Sigma$, truncating all infinite terms of λ to a high enough number produces another element of Σ. Moreover, if $\lambda \in \Sigma$ is minimal, we must have $\lambda_1 = \lambda_2 = \cdots = \lambda_{r-k}$. If this were not the case, we could consider the partition λ' such that $\lambda'_1 = \cdots = \lambda'_{r-k} = \lambda_{r-k}$, and $\lambda'_i = \lambda_i$ for $i > r-k$. Then λ' would also be in Σ, but $\lambda' \prec \lambda$, contradicting the fact that λ is minimal. It is also clear that minimal elements of Σ must verify $\lambda_{r-k} + \cdots + \lambda_r = p$. In fact, if a partition in Σ does not verify this, we can decrease the last terms of the partition an still remain in Σ.

So far we know that the minimal elements in Σ are partitions that verify $\lambda_1 = \cdots = \lambda_{r-k}$ and $\lambda_{r-k} + \cdots + \lambda_r = p$. Note that we assume $0 < k < r-1$, so for any two partitions λ, λ' with the previous properties, if $\lambda_{r-k} \neq \lambda'_{r-k}$, then λ and λ' are not comparable.

Pick a minimal element $\lambda \in \Sigma$, and write $d = \lambda_{r-k}$. The proposition will follow if we show that the sequence $(\lambda_{r-k}, \lambda_{r-k+1}, \ldots, \lambda_\ell)$ is of the form $(d, \ldots, d, e)$ for some $0 \leq e < d$. But this is clear from the analysis of the covering relations given by Proposition 3.3.7. Consider the Young diagram Γ associated to λ. The longest row of Γ has length d. If there are two rows, say $i < j$, with length less than d, then we must have $r - k < i$ and we can move one box from row j to row i and obtain a partition still in Σ but subordinate to λ. This contradicts the fact that λ is minimal, and we see that λ must have the form given in the proposition. $\square$

Proposition 3.4.4. *Assume that $k = 0$ or $k = r - 1$. Then $\mathrm{Cont}^p(D^k) \subset M_\infty$ is irreducible and contains a unique dense orbit $\mathcal{C}_\lambda$, where $\lambda = (p^{r-k})$.*

Proof. Form Proposition 3.2.6, the orbits $\mathcal{C}_\lambda$ contained in $\mathrm{Cont}^p(D^0)$ are the ones that verify $\lambda_r \geq p$. It is clear that the minimal partition of this type is (p^r). Analogously, $\mathrm{Cont}^p(D^{r-1})$ contains orbits whose associated partitions verify $\lambda_1 + \cdots + \lambda_r \geq p$, and the minimal one among these is (p^1). □

Theorem 3.4.5. *If $k = 0$ or $k = r - 1$, the contact locus $\mathrm{Cont}^p(D^k) \subset M_\infty$ is irreducible. Otherwise, the number of irreducible components of $\mathrm{Cont}^p(D^k) \subset M_\infty$ is*

$$p + 1 - \left\lceil \frac{p}{k+1} \right\rceil .$$

Proof. The first assertion follows directly from Proposition 3.4.4. For the second one, we need to count the number of partitions that appear in Proposition 3.4.2. Recall that these were partitions of the form $\lambda^d = (d^{a+r-k}, e^1)$ of length at most r such that $p = (a+1)d + e$ and $0 \leq e < d$. Since d ranges from 0 to p, we have at most $p + 1$ such partitions. But as we decrease d, the length of λ^d increases, possibly surpassing the limit r. Therefore the number of allowed partitions is $p + 1 - d_0$, where d_0 is the smallest integer such that λ^{d_0} has length no greater than r.

If d divides p, the length of λ^d is $(\frac{p}{d} - 1 + r - k)$. Otherwise it is $(\lfloor \frac{p}{d} \rfloor + r - k)$. In either case, the length is no greater that r if and only if $d \geq \lceil \frac{p}{k+1} \rceil$. Hence $d_0 = \lceil \frac{p}{k+1} \rceil$, and the theorem follows. □

Corollary 3.4.6. *It $k = 0$ or $k = r - 1$, the jet scheme D^k_n is irreducible. Otherwise, the number of irreducible components of D^k_n is*

$$n + 2 - \left\lceil \frac{n+1}{k+1} \right\rceil .$$

Proof. The contact locus $\mathrm{Cont}^{n+1}(D^k)$ is the inverse image of the jet scheme D^k_n under the truncation map $M_\infty \to M_n$. Since M is smooth, this truncation map is surjective, so D^k_n has the same number of components as $\mathrm{Cont}^{n+1}(D^k)$. Now the result follows directly from Theorem 3.4.5. □

Remark 3.4.7 (Nash's problem)**.** We end this section with a discussion of Nash's problem in the case of generic determinantal varieties. For a variety X with singular locus S, the Nash map associates to each fat irreducible component of $\mathrm{Cont}^1(S) \subset X_\infty$ an exceptional divisor in every resolution of singularities of X. Nash conjectured that this should give a bijection between fat irreducible components and "essential" divisors (divisors appearing in any resolution of singularities). This conjecture has been shown to be false in general (Ishii and Kollár, 2003), but it is still an interesting problem to understand the class of singularities for which the conjecture holds. Recall that D^k is singular when $0 < k < r$, and that its singular locus is D^{k-1}. The blowing-up of D^k along D^{k-1} is smooth, and this gives a resolution of D^k having irreducible exceptional locus (Arbarello et al., 1985, §II.2). This implies that Nash's conjecture holds for D^k, and that there should be a unique irreducible component in $\mathrm{Cont}^1(D^{k-1}) \cap D^k_\infty$. This agrees with our computations. When translated into the language of partitions, the problem of counting components in $\mathrm{Cont}^1(D^{k-1}) \cap$

D^k_∞ is equivalent to counting components in $\mathrm{Cont}^1(D^{r-1})$ after replacing r with k. But Theorem 3.4.5 tells us that this contact locus is irreducible.

3.5 Discrepancies and log canonical thresholds

In this section we compute discrepancies for all invariant divisorial valuations over M and over D^k, and use it to give formulas for log canonical thresholds involving determinantal varieties. We start with a proposition that determines all possible invariant maximal divisorial sets in terms of orbits in the arc space.

Proposition 3.5.1 (Divisorial sets = Orbit closures, M)**.** *Let ν be a G-invariant divisorial valuation over M, and let $\mathcal{C}$ be the associated maximal divisorial set in M_∞. Then there exists a unique partition $\lambda \in \Lambda_r$ of length at most r whose associated orbit $\mathcal{C}_\lambda$ is dense in $\mathcal{C}$. Conversely, the closure of $\mathcal{C}_\lambda$, where λ is a partition, is a maximal divisorial set associated to an invariant valuation.*

Proof. Recall from Section 2.2 (or see (Ishii, 2008)) that $\mathcal{C}$ is the union of the fat sets of M_∞ that induce the valuation ν. Therefore, since ν is G-invariant, $\mathcal{C}$ is G_∞-invariant and can be written as a union of orbits. Note that the thin orbits of M_∞ are all contained in D^{r-1}_∞, and that $\mathcal{C}$ is itself fat, so $\mathcal{C}$ must contain a fat orbit. Let $\Sigma \subset \Lambda_r$ be the set of partitions indexing fat orbits contained in M. For $\mu \in \Sigma$ we denote by ν_μ the valuation induced by $\mathcal{C}_\mu$. Then, for $f \in \mathcal{O}_M$ we have:

$$\nu(f) \quad = \quad \min_{\gamma\in\mathcal{C}} \{ \mathrm{ord}_\gamma(f) \} \quad = \quad \min_{\mu\in\Sigma} \min_{\gamma\in\mathcal{C}_\mu} \{ \mathrm{ord}_\gamma(f) \} \quad = \quad \min_{\mu\in\Sigma} \{ \nu_\mu(f) \}.$$

As a consequence, since ν_μ is determined by its value on the ideals $\mathcal{I}_{D^0},\ldots,\mathcal{I}_{D^{r-1}}$, the same property holds for ν. Let $\lambda = (\lambda_1,\ldots,\lambda_r)$ be such that $\nu(\mathcal{I}_{D^k}) = \lambda_r + \cdots + \lambda_{r-k}$. From the fact that $\mathcal{I}_{D^k}\,\mathcal{I}_{D^{k-2}} \subset \mathcal{I}^2_{D^{k-1}}$ we deduce that $\lambda_k \geq \lambda_{k+1}$, and we get a partition $\lambda \in \Lambda_r$ whose associated orbit $\mathcal{C}_\lambda$ induces the valuation ν (so $\lambda \in \Sigma$). The proposition follows if we show that $\mathcal{C}_\lambda$ is dense in $\mathcal{C}$.

Consider $\mu \in \Sigma$. Since $\mathcal{C}_\mu \subset \mathcal{C}$, we know that $\nu_\mu \geq \nu$, and we get that

$$\mu_r + \cdots + \mu_{r-k} \;=\; \nu_\mu(\mathcal{I}_{D^k}) \;\geq\; \nu(\mathcal{I}_{D^k}) \;=\; \lambda_r + \cdots + \lambda_{r-k}.$$

Hence $\lambda \preceq \mu$, and Theorem 3.3.11 tells us that $\mathcal{C}_\mu$ is contained in the closure of $\mathcal{C}_\lambda$, as required. $\square$

Notation 3.5.2. For the purpose of the next proposition it will be convenient to introduce the following notation. Fix positive integers $k < r$. Given a partition $\lambda = (\lambda_1,\ldots,\lambda_\ell) \in \Lambda_k$ of length at most k, denote by $\lambda^+ = (\infty,\ldots,\infty,\lambda_1,\ldots,\lambda_\ell) \in \overline{\Lambda}_r$ the extended partition obtained by adjoining $r-k$ infinities.

Proposition 3.5.3 (Divisorial sets = Orbit closures, D^k)**.** *Let ν be a G-invariant divisorial valuation over D^k, and let $\mathcal{C}$ be the associated maximal divisorial set in D^k_∞. Then there exists a unique partition $\lambda \in \Lambda_k$ such that the orbit $\mathcal{C}_{\lambda^+}$ is dense in $\mathcal{C}$. Conversely, the closure of $\mathcal{C}_{\lambda^+}$, where $\lambda \in \Lambda_k$, is a maximal divisorial set in D^k_∞ associated to a G-invariant divisorial valuation.*

Proof. Analogous to the proof of 3.5.1. $\square$

We now proceed to compute discrepancies for invariant divisorial valuations. These are closely related to the codimensions of the corresponding maximal divisorial sets, which by the previous propositions are just given by orbit closures. Since orbits are cylinders, their codimension can be computed by looking at the corresponding orbit in a high enough jet scheme. But jet schemes are of finite type, so orbits have a finite dimension that can be computed via the codimension of the corresponding stabilizer. For this reason, we will try to understand the structure of the different stabilizers in the jet schemes G_n.

Recall from Definition 3.2.4 that $\mathcal{C}_\lambda$ is the orbit containing the following matrix:

$$\delta_\lambda = \begin{pmatrix} 0 & \cdots & 0 & t^{\lambda_1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & t^{\lambda_2} & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & t^{\lambda_r} \end{pmatrix}.$$

This matrix defines an element of the jet scheme M_n as long as n is greater than the highest term of λ; the corresponding G_n-orbit in M_n is denoted by $\mathcal{C}_{\lambda,n}$. The following proposition determines the codimension of the stabilizer of δ_λ in the jet group G_n.

Proposition 3.5.4. *Let $\lambda \in \Lambda_r$ be a partition of length at most r, and let n be a positive integer greater than the highest term of λ. Let $H_{\lambda,n}$ denote the stabilizer of δ_λ in the group G_n. Then*

$$\operatorname{codim}(H_{\lambda,n}, G_n) = (n+1)rs - \sum_{i=1}^{r} \lambda_i(s - r + 2i - 1).$$

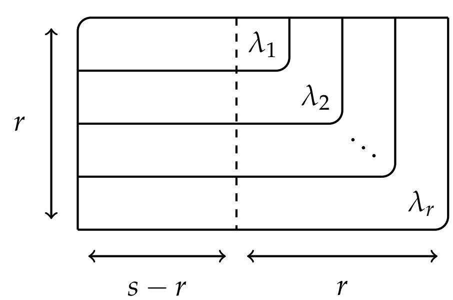
Proof. Pick $(g,h) \in G_n = (\mathrm{GL}_r)_n \times (\mathrm{GL}_s)_n$. Then:

$$(g,h) \in H_{\lambda,n} \qquad \Leftrightarrow \qquad g \cdot \delta_\lambda \cdot h^{-1} = \delta_\lambda \qquad \Leftrightarrow \qquad g \cdot \delta_\lambda = \delta_\lambda \cdot h \qquad \Leftrightarrow$$

$$\begin{pmatrix} 0 & \cdots & 0 & t^{\lambda_1}* & t^{\lambda_2}* & \cdots & t^{\lambda_r}* \\ 0 & \cdots & 0 & t^{\lambda_1}* & t^{\lambda_2}* & \cdots & t^{\lambda_r}* \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & t^{\lambda_1}* & t^{\lambda_2}* & \cdots & t^{\lambda_r}* \end{pmatrix} = \begin{pmatrix} t^{\lambda_1}* & \cdots & t^{\lambda_1}* & t^{\lambda_1}* & t^{\lambda_1}* & \cdots & t^{\lambda_1}* \\ t^{\lambda_2}* & \cdots & t^{\lambda_2}* & t^{\lambda_2}* & t^{\lambda_2}* & \cdots & t^{\lambda_2}* \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ t^{\lambda_r}* & \cdots & t^{\lambda_r}* & t^{\lambda_r}* & t^{\lambda_r}* & \cdots & t^{\lambda_r}* \end{pmatrix}.$$

This equality of matrices gives one equation of the form $t^{a(i,j)}* = t^{b(i,j)}*$ for each entry (i,j) in a $r \times s$ matrix. We have $a(i,j) = \lambda_{j-s+r}$ and $b(i,j) = \lambda_i$ (assume $\lambda_j = \infty$ for $j < 0$).

Each equation of the form $t^a * = t^b *$ gives $(n+1) - \min\{a,b\}$ independent equations on the coefficients of the power series, so it reduces the dimension of the stabilizer by $(n+1) - \min\{a,b\}$. The entries (i,j) for which $\min\{a(i,j), b(i,j)\} = \lambda_k$ form an $\lrcorner$-shaped region of the $r \times s$ matrix, as we illustrate in the following diagram:



The region corresponding to λ_i contains $(s - r + 2i - 1)$ entries, and the result follows. $\square$

Proposition 3.5.5. *Let $\lambda \in \overline{\Lambda}_r$ be an extended partition of length at most r, and consider its associated G_∞-orbit $\mathcal{C}_\lambda$ in M_∞. If λ contains infinite terms, $\mathcal{C}_\lambda$ has infinite codimension. If λ is a partition, the codimension is given by:*

$$\operatorname{codim}(\mathcal{C}_\lambda, M_\infty) = \sum_{i=1}^{r} \lambda_i (s - r + 2i - 1).$$

Proof. If λ contains infinite terms, $\mathcal{C}_\lambda$ is thin, so it has infinite codimension. Otherwise Proposition 3.2.11 tells us that $\mathcal{C}_\lambda$ is the inverse image of $\mathcal{C}_{\lambda,n}$ under the truncation map $M_\infty \to M_n$ for n large enough. Since M is smooth, we see that the codimension of $\mathcal{C}_\lambda$ in M_∞ is the same as the codimension of $\mathcal{C}_{\lambda,n}$ in M_n. The dimension of $\mathcal{C}_{\lambda,n}$ is the codimension of the stabilizer of δ_λ in G_n. The result now follows from Proposition 3.5.4 and the fact that M_n has dimension $(n+1)rs$. □

Corollary 3.5.6. *Let ν be a G-invariant valuation of M and let $\lambda \in \Lambda_r$ be the unique partition such that $\mathcal{C}_\lambda$ induces ν. Let $k_\nu(M)$ be the discrepancy of M along ν, and let q_ν be the multiplicity of ν. Then*

$$k_\nu(M) + q_\nu = \sum_{i=1}^{r} \lambda_i (s - r + 2i - 1).$$

Proof. From Proposition 3.5.1 we know that the closure of $\mathcal{C}_\lambda$ is the maximal divisorial set associated to ν. Since M is smooth, the log discrepancy $k_\nu(M) + q_\nu$ agrees with the codimension of the associated maximal divisorial set (see Section 2.3). The result now follows from Proposition 3.5.5. □

Theorem 3.5.7. *Recall that M denotes the space of matrices of size $r \times s$ and D^k is the variety of matrices of rank at most k. The log canonical threshold of the pair (M, D^k) is*

$$\operatorname{lct}(M, D^k) = \min_{i=0\ldots k} \frac{(r-i)(s-i)}{k+1-i}.$$

Proof. We will use Mustaţă's formula (see (Ein et al., 2004, Cor. 3.2)) to compute log canonical thresholds:

$$\operatorname{lct}(M, D^k) = \min_n \left\{ \frac{\operatorname{codim}(D^k_n, M_n)}{n+1} \right\} = \min_p \left\{ \frac{\operatorname{codim}(\operatorname{Cont}^p(D^k), M_\infty)}{p} \right\}.$$

Let $\Sigma_p \subset \overline{\Lambda}_r$ be the set of extended partitions of length at most r such that $\lambda_r + \cdots + \lambda_{r-k} = p$. By Propositions 3.2.6 and 3.4.2, we have:

$$\operatorname{lct}(M, D^k) = \min_p \min_{\lambda \in \Sigma_p} \left\{ \frac{\operatorname{codim}(\mathcal{C}_\lambda, M_\infty)}{p} \right\}.$$

Consider the following linear function

$$\psi(a_1, \ldots, a_r) = \sum_{i=1}^{r} a_i (s - r + 2i - 1).$$

Then, by Proposition 3.5.5, we get:

$$\operatorname{lct}(M, D^k) = \min_p \min_{\lambda \in \Sigma_p} \left\{ \frac{\psi(\lambda)}{p} \right\} = \min_p \min_{\lambda \in \Sigma_p} \left\{ \psi\left(\tfrac{\lambda}{p}\right) \right\}.$$

Let $\Sigma \subset \mathbf{Q}^r$ be the set of tuples $(a_1, \ldots, a_r)$ such that $a_1 \geq a_2 \geq \cdots \geq a_r \geq 0$ and $a_r + \cdots + a_{r-k} = 1$. Then:

$$\operatorname{lct}(M, D^k) = \min_{a \in \Sigma} \{\psi(a)\}.$$

The map $\varphi(a_1, \ldots, a_r) = (a_1 - a_2, \ldots, a_{r-1} - a_r, a_r)$ sends Σ to Σ', where $\Sigma' \subset \mathbf{Q}^r$ is the set of tuples $(b_1, \ldots, b_r)$ such that $b_i \geq 0$ and $(k+1)b_r + kb_{r-1} \cdots + b_{r-k} = 1$. Then

$$\operatorname{lct}(M, D^k) = \min_{b \in \Sigma'} \{\xi(b)\},$$

where

$$\xi(b) = \psi(\varphi^{-1}(b)) = \sum_{i=1}^{r} (b_r + b_{r-1} + \cdots + b_i)(s - r + 2i - 1) = \sum_{j=1}^{r} b_j \, j \, (s - r + j).$$

Note that in the definition of Σ' the only restriction on the first $r - k - 1$ coordinates $b_1, b_2, \ldots, b_{r-k-1}$ is that they are nonnegative. Let Σ'' be the subset of Σ' obtained by setting $b_1 = \cdots = b_{r-k-1} = 0$. From the formula for $\xi(b)$ we see that the minimum $\min_{b \in \Sigma'}\{\xi(b)\}$ must be achieved in Σ''. But Σ'' is a simplex and ξ is linear, so the minimum is actually achieved in one of the extremal points of Σ''. These extremal points are:

$$P_{r-k} = (0, \ldots, 0, 1, 0, \ldots, 0, 0), \quad P_{r-k+1} = (0, \ldots, 0, 0, \tfrac{1}{2}, \ldots, 0, 0), \quad \ldots$$
$$\ldots \quad P_{r-1} = (0, \ldots, 0, 0, 0, \ldots, \tfrac{1}{k}, 0), \quad P_r = (0, \ldots, 0, 0, 0, \ldots, 0, \tfrac{1}{k+1}).$$

The value of ξ at these points is:

$$\xi(P_{r-i}) = \frac{1}{k+1-i}(r-i)(s-i).$$

Therefore

$$\mathrm{lct}(M, D^k) = \min_{i=0\dots k} \frac{(r-i)(s-i)}{k+1-i},$$

as required. $\square$

3.6 Some motivic integrals

In the previous section we computed codimensions of orbits in the arc space M_∞, as a mean to obtain formulas for discrepancies and log canonical thresholds. But a careful look at the proofs shows that we can understand more about the orbits than just their codimensions. As an example of this, in this section we compute the motivic volume of the orbits in the arc space.

Throughout this section, we will restrict ourselves to the case of square matrices, i.e. we assume $r = s$.

Before we state the main proposition, we need to recall some notions from the group theory of GL_r: parabolic subgroups, Levi factors, flag manifolds, and the natural way to obtain a parabolic subgroup from a partition.

Definition 3.6.1. Let $0 < v_1 < v_2 < \cdots < v_j < r$ be integers. A *flag* in $\mathbf{C}^r$ of signature $(v_1, \dots, v_j)$ is a nested chain $V_1 \subset V_2 \subset \cdots \subset V_j \subset \mathbf{C}^r$ of vector subspaces with

$\dim V_i = v_i$. The general linear group GL_r acts transitively on the set of all flags with a given signature. The stabilizer of a flag is known as a *parabolic subgroup* of GL_r. If $P \subset \mathrm{GL}_r$ is a parabolic subgroup, the quotient GL_r / P parametrizes flags of a given signature and it is known as a *flag variety*.

Definition 3.6.2. Let $\{e_1, \ldots, e_r\}$ be the standard basis for $\mathbf{C}^r$, and let $\lambda = (d_1^{a_1} \ldots d_j^{a_j}) \in \Lambda_r$ be a partition. Write $a_{j+1} = r - \sum_{i=1}^{j} a_i$ and $v_i = a_1 + \cdots + a_i$, and consider the following vector subspaces of $\mathbf{C}^r$:

$$V_i = \mathrm{span}(e_1, \ldots, e_{v_i}), \qquad\qquad W_i = \mathrm{span}(e_{v_{i-1}+1}, \ldots, e_{v_i}).$$

We denote by P_λ the stabilizer of the flag $V_1 \subset \cdots \subset V_j$ and call it the *parabolic subgroup of* GL_r *associated to* λ. The group $L_\lambda = \mathrm{GL}_{a_1} \times \cdots \times \mathrm{GL}_{a_{j+1}}$ embeds naturally in P_λ as the group endomorphisms of W_i, and it is known as the *Levi factor* of the parabolic P_λ.

Example 3.6.3. Assume $r = 6$ and consider the partition $\lambda = (4,4,4,1,1) = (4^3 1^2)$. Then P_λ and L_λ are the groups of invertible $r \times r$ matrices of the forms

$$P_\lambda : \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}, \qquad L_\lambda : \begin{pmatrix} * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}.$$

Proposition 3.6.4. *Assume that $r = s$. Let $\lambda \in \Lambda_r$ be a partition of length at most r and consider its associated parabolic subgroup P_λ and Levi factor L_λ. Let μ be the motivic measure in M_∞, and $\mathcal{C}_\lambda$ the orbit in M_∞ associated to λ. If b is the log discrepancy of the valuation induced by $\mathcal{C}_\lambda$, we have:*

$$\mu(\mathcal{C}_\lambda) \quad = \quad \mathbf{L}^{-b}\,[\mathrm{GL}_r\,/P_\lambda]^2\,[L_\lambda].$$

Proof. Consider n, δ_λ and $H_{\lambda,n} \subset G_n$ as in Proposition 3.5.4. If $\mathcal{C}_{\lambda,n}$ is the truncation of $\mathcal{C}_\lambda$ to M_n, we know that for n large enough

$$\mu(\mathcal{C}_\lambda) \;=\; \mathbf{L}^{-r^2n}\,[\mathcal{C}_{\lambda,n}] \;=\; \mathbf{L}^{-r^2n}\,[G_n]\,[H_{\lambda,n}]^{-1} \;=\; \mathbf{L}^{r^2n}\,[\mathrm{GL}_r]^2\,[H_{\lambda,n}]^{-1}.$$

At the beginning of the proof of Proposition 3.5.4 we found the equations defining $H_{\lambda,n}$:

$$(g,h) \in H_{\lambda,n} \qquad \Leftrightarrow \qquad g\cdot\delta_\lambda\cdot h^{-1} = \delta_\lambda \qquad \Leftrightarrow \qquad g\cdot\delta_\lambda = \delta_\lambda\cdot h \qquad \Leftrightarrow$$

$$\begin{pmatrix} t^{\lambda_1}* & t^{\lambda_2}* & \cdots & t^{\lambda_r}* \\ t^{\lambda_1}* & t^{\lambda_2}* & \cdots & t^{\lambda_r}* \\ \vdots & \vdots & \ddots & \vdots \\ t^{\lambda_1}* & t^{\lambda_2}* & \cdots & t^{\lambda_r}* \end{pmatrix} = \begin{pmatrix} t^{\lambda_1}* & t^{\lambda_1}* & \cdots & t^{\lambda_1}* \\ t^{\lambda_2}* & t^{\lambda_2}* & \cdots & t^{\lambda_2}* \\ \vdots & \vdots & \ddots & \vdots \\ t^{\lambda_r}* & t^{\lambda_r}* & \cdots & t^{\lambda_r}* \end{pmatrix}. \tag{3.1}$$

As a variety, G_n can be written as product $G \times \mathfrak{g}^n$, where $\mathfrak{g} \simeq \mathbf{A}^{2r^2}$ is the Lie algebra of G. For n large enough, the equations of $H_{\lambda,n}$ above are compatible with this product structure, in the sense that $H_{\lambda,n} = H \times \mathfrak{h}$, for some subgroup $H \subset G$ and some subvariety

$\mathfrak{h} \subset \mathfrak{g}^n$. The structure of $\mathfrak{h}$ is very simple. If $\{z_1, \ldots, z_{2nr^2}\}$ are coordinates for $\mathfrak{g}^n \simeq \mathbf{A}^{2nr^2}$, the ideal of $\mathfrak{h}$ in $\mathfrak{g}^n$ is generated by some subset of these coordinates, i.e. by $\{z_\sigma \,|\, \sigma \in \Omega\}$ for some $\Omega \subset \{1, \ldots, 2nr^2\}$. The cardinality of Ω can be computed with the same method used in the proof of Proposition 3.5.4:

$$\operatorname{codim}(\mathfrak{h}, \mathfrak{g}^n) = \#\Omega = nr^2 - \sum_{i=1}^{r} \lambda_i(2i-1) = nr^2 - b,$$

where b is the log discrepancy of the valuation induced by $\mathcal{C}_\lambda$. As a consequence

$$[\mathfrak{h}] \quad = \quad [\mathfrak{g}^n]\, \mathbf{L}^{-nr^2+b} \quad = \quad \mathbf{L}^{nr^2+b},$$

and

$$\mu(\mathcal{C}_\lambda) \;=\; \mathbf{L}^{r^2 n}\, [\mathrm{GL}_r]^2\, [H_{\lambda,n}]^{-1} \;=\; \mathbf{L}^{-b}\, [\mathrm{GL}_r]^2\, [H]^{-1}.$$

Let $g_{i,j}$ and $h_{i,j}$ be the natural coordinates on $G = \mathrm{GL}_r \times \mathrm{GL}_r$. By (Equation 3.1), the subgroup H is given by

$$g_{i,j} = h_{i,j} \qquad \text{if } \lambda_i = \lambda_j, \tag{3.2}$$

$$g_{i,j} = 0 \qquad \text{if } \lambda_i \neq \lambda_j \text{ and } i < j, \tag{3.3}$$

$$h_{i,j} = 0 \qquad \text{if } \lambda_i \neq \lambda_j \text{ and } i > j. \tag{3.4}$$

Form (Equation 3.3) and (Equation 3.4), we see that H is a subgroup of $P_\lambda^{\mathrm{op}} \times P_\lambda \subset G$, and (Equation 3.2) tells us that we can obtain H from $P_\lambda^{\mathrm{op}} \times P_\lambda$ by identifying the two copies of the Levi L_λ. Hence $[H] = [P_\lambda]^2 \, [L_\lambda]^{-1}$ and the theorem follows. $\square$

CHAPTER 4

ARCS ON QUASI-HOMOGENEOUS SPACES

In this chapter we investigate possible generalizations of our results on generic determinantal varieties to a more general setting.

The main ingredient of our analysis in Chapter 3 was the presence of a group action whose orbits in the arc space we were able to understand very well. This essentially reduced the problem of computing jet schemes, discrepancies and motivic volumes to combinatorics in a poset. For generic determinantal varieties, this poset happened to be isomorphic to a space of partitions, an extensively studied object.

It is natural to ask what were the properties of the action that allowed us to carry out the computations. Looking at simple examples, one soon realizes that the study of a general action is hopeless; one needs a simple enough action so that we can classify not only orbits in the variety but also in its arc space. In the first two sections of the chapter we formalize this idea. We show that the problem of classifying orbits in the arc space is essentially equivalent to a problem of classifying valuations of a certain simple type. We then single out spherical varieties as those varieties for which this classification problem is manageable.

Once we understand the orbit decomposition to a certain extent, the next step is to determine its poset structure. The remainder of the chapter investigates this problem in a few specific examples. We first briefly recall the structure of the arc space of toric varieties, and see how the results in this situation resemble the case of generic determinantal varieties. Then we analyze another class of determinantal varieties, namely the ones living in the space of skew-symmetric matrices. Finally we extend the results for toric varieties to determine the poset structure in the set of orbits of the arc space of a toroidal embedding of a symmetric space.

4.1 Invariant valuations and orbits in the arc space

The main objective of this section is to prove an analogue of Proposition 3.5.1 for an arbitrary quasi-homogeneous space. This is essentially a result due to Luna and Vust (Luna and Vust, 1983), which we adapt here to the language of arc spaces.

Definition 4.1.1. Let G be an algebraic group and X a normal algebraic variety with a rational action by G. If the action is transitive, we say that X is a *homogeneous space*. Equivalently, a homogeneous space is isomorphic to a quotient G/H, where H is a closed subgroup of G. If X contains a dense orbit we say that X is a *quasi-homogeneous* space. Given a homogeneous space G/H, an *equivariant embedding* of G/H is a quasi-homogeneous space X with the choice of an equivariant isomorphism of the dense orbit of X with G/H. Notice that such an isomorphism is determined by the choice of a point in the dense orbit of X.

We will study orbits in the arc space of an equivariant embedding of a homogeneous space, and relate them to invariant valuations. As we will see, the answer is in some way independent of the choice of embedding. We can make this precise by introducing the notion of loop space.

Definition 4.1.2 (Loop space)**.** Given a scheme X, the *loop space* of X is the set of $\mathbf{C}((t))$-valued points of X. We denote it by $\mathcal{L}(X)$.

$$\mathcal{L}(X) = \mathrm{Hom}_{\mathbf{C}\text{-Sch}}(\mathrm{Spec}\,\mathbf{C}((t)), X)$$

Remark 4.1.3. If X is separated, the valuative criterion for separated morphisms gives an inclusion of the arc space in the loop space, $X_\infty \subset \mathcal{L}(X)$. Moreover, the valuative criterion for properness guarantees that this containment is a bijection if and only if X is proper.

Remark 4.1.4. As in the case of the arc space, the assignment $X \to \mathcal{L}(X)$ is functorial. This guarantees that $\mathcal{L}(G)$ is a group whenever G is an algebraic group, and that any action of G on a variety X induces an action of $\mathcal{L}(G)$ on $\mathcal{L}(X)$. Moreover, the natural inclusion of groups $G_\infty \subset \mathcal{L}(G)$ induces an action of G_∞ on $\mathcal{L}(X)$, where X is any G-variety.

Remark 4.1.5. As opposed to the arc space, the loop space is not a scheme whenever X is not proper. This will not cause problems for us, as we are only interested in enumerating orbits in the loop space. For our purposes, $\mathcal{L}(X)$ is just a set.

The main reason to introduce loop spaces is the following proposition, which effectively characterizes the orbits giving rise to valuations as those that live in the loop space of G/H.

Proposition 4.1.6. *Let X be an equivariant embedding of G/H. Both X_∞ and $\mathcal{L}(G/H)$ are contained in $\mathcal{L}(X)$, so we can consider their intersection $X_\infty \cap \mathcal{L}(G/H)$. The union of the fat G_∞-orbits of X_∞ is $X_\infty \cap \mathcal{L}(G/H)$.*

Proof. Let $\mathcal{C}$ denote a G_∞-orbit in X_∞. Let Z be the complement of G/H in X. Then by definition $X_\infty \setminus Z_\infty = \mathcal{L}(G/H) \cap X_\infty$. Since Z is G-invariant, its arc space Z_∞ is also G_∞-invariant, and $\mathcal{C}$ is either contained in Z_∞ or in $\mathcal{L}(G/H)$. In particular, if $\mathcal{C}$ is fat it must be contained in $\mathcal{L}(G/H)$.

Conversely, assume that $\mathcal{C}$ is contained in $\mathcal{L}(G/H)$. Let α denote the generic point of $\mathcal{C}$, and η the generic point of $\operatorname{Spec} \mathbf{C}[\![t]\!]$. Then the point $\alpha(\eta) \in X$ is G-invariant (because $\mathcal{C}$ is a G_∞-orbit) and contained in G/H (since $\mathcal{C} \subset \mathcal{L}(G/H)$), hence it must be the generic point of X. Any closed subscheme of X containing $\mathcal{C}$ in its arc space must contain the point $\alpha(\eta)$; therefore the only such subscheme is the whole space X, and we see that $\mathcal{C}$ is fat. $\square$

Theorem 4.1.11 below will identify orbits in $\mathcal{L}(G/H)$ with a class of invariant valuations. Unfortunately, one cannot hope in general to express each invariant valuation in terms of a single orbit. The reason behind this is simple: if one can find a divisor in an equivariant embedding of G/H containing a continuous family of orbits, the corresponding valuation will give rise to a cylinder in the arc space which does not contain a dense orbit. The following definition excludes this types of valuations.

Definition 4.1.7 (Simple invariant valuations)**.** Let ν be an invariant valuation of G/H. We say that ν is *divisorial* if there exists an equivariant embedding X of G/H and an invariant divisor $D \subset X$ such that $\nu = q\,\mathrm{val}_D$, where val_D is the valuation induced by D and q is a positive integer. A divisorial invariant valuation ν of G/H is said to be *simple* if the corresponding divisor D contains a dense G-orbit. The set of simple invariant valuations of G/H is denoted by $\mathcal{V}_s(G/H)$.

Remark 4.1.8. The condition on the divisor in the previous definition can be expressed directly in terms of the valuation, without references to any model. Given an invariant divisorial valuation ν over G/H, let k_ν be its residue field. Then ν is simple if and only if $k_\nu^G = \mathbf{C}$, where k_ν^G denotes the subfield of G-invariants of k_ν.

We need one more definition before we can state the main theorem of this section.

Definition 4.1.9. The group $\mathrm{Aut}_{\mathbf{C}}\,\mathbf{C}[\![t]\!]$ of $\mathbf{C}$-algebra automorphisms of $\mathbf{C}[\![t]\!]$ acts naturally on $\mathrm{Spec}\,\mathbf{C}[\![t]\!]$ and $\mathrm{Spec}\,\mathbf{C}(\!(t)\!)$, and therefore on the arc space X_∞ and the loop space $\mathcal{L}(X)$ of any scheme X. In particular, if G is an algebraic group, we can use this action to form the semi-direct product $G_\infty \rtimes \mathrm{Aut}_{\mathbf{C}}\,\mathbf{C}[\![t]\!]$; we denote it by G_∞^+. If G acts on an scheme X, we obtain an action of G_∞^+ on X_∞ and on $\mathcal{L}(X)$.

Remark 4.1.10. It is easy to check that $\mathrm{Aut}_{\mathbf{C}}\,\mathbf{C}[\![t]\!]$ is a group scheme (not of finite type), and that its action on X_∞ is rational. In particular G_∞^+ is also a group scheme acting rationally on X_∞ for any scheme X on which G acts rationally.

Theorem 4.1.11 ((Luna and Vust, 1983, Prop. in §4.10)). *There is a one to one correspondence between the set of G_∞^+-orbits of the loop space $\mathcal{L}(G/H)$ and the set of simple invariant valuations over G/H.*

$$\mathcal{V}_s(G/H) \simeq \mathcal{L}(G/H)/G_\infty^+.$$

Remark 4.1.12. In view of Proposition 4.1.6, we can rephrase the statement of the previous theorem as follows: given an equivariant embedding X, there is a one to one correspondence between fat G_∞^+ orbits of X_∞ and simple invariant valuations whose center is defined on X.

The rest of the section will be devoted to the proof of Theorem 4.1.11. We first need to define a map form orbits in $\mathcal{L}(G/H)$ to valuations of $\mathbf{C}(G/H)$.

Definition 4.1.13. Let $\mathcal{C}$ be a G_∞-orbit in $\mathcal{L}(G/H)$. Pick an equivariant embedding X of G/H such that $\mathcal{C}$ is contained in X_∞ (for example, pick X proper). Since $\mathcal{C}$ is fat in X_∞, it defines a valuation of $\mathbf{C}(X) = \mathbf{C}(G/H)$, which we denote by $\nu_{\mathcal{C}}$. This valuation is independent of the choice of embedding X, and we call it the valuation *associated* to the orbit $\mathcal{C}$.

It is clear that the valuation associated to an orbit in $\mathcal{L}(G/H)$ is invariant. We will now show that it is also divisorial and simple. For this we first need the following two technical lemmas, taken from (Luna and Vust, 1983).

Lemma 4.1.14 (Lifting loops to the group). *Let $\lambda \in \mathcal{L}(G/H)$ a loop in in G/H. Given a positive integer q, let λ^q be the loop obtained by precomposing with the twist $t \mapsto t^q$. Then* *there exist a loop $\mu \in \mathcal{L}(G)$ and a positive integer q such that μ maps to λ^q via the natural map $\mathcal{L}(G) \to \mathcal{L}(G/H)$.*

Proof. See (Luna and Vust, 1983, Lemma in §4.3). □

Lemma 4.1.15 (Orbits contain germs of curves)**.** *Let $\mathcal{C}$ be a G_∞-orbit in $\mathcal{L}(G)$. Then $\mathcal{C}$ contains the germ of a map from a punctured curve. More precisely, there exist a smooth algebraic curve C, a point $c \in C$ and a map $\gamma : C \setminus \{c\} \to G$ such that the induced loop $\gamma_c : \operatorname{Spec} \mathbf{C}((t)) \to G$ is contained in $\mathcal{C}$.*

Proof. See (Luna and Vust, 1983, Lemma in §4.5). □

Proposition 4.1.16 (Orbits induce divisorial valuations)**.** *For each G_∞-orbit $\mathcal{C}$ in $\mathcal{L}(G/H)$, the associated valuation $\nu_{\mathcal{C}}$ is divisorial.*

Proof. Let λ be a loop in $\mathcal{C}$. The transformation $\lambda(t) \mapsto \lambda(t^n)$ amounts to changing the multiplicity of the valuation $\nu_{\mathcal{C}}$, and this causes no problem as we are only interested in determining whether $\nu_{\mathcal{C}}$ is divisorial or not. Hence we can use Lemma 4.1.14 and assume that λ is of the form $\mu \cdot H/H$ for some loop $\mu \in \mathcal{L}(G)$. Furthermore, from Lemma 4.1.15 we can choose λ in such a way that μ is the germ of some (punctured) curve C mapping into G. We obtain a dominant map

$$G \times (C \setminus \{c\}) \longrightarrow G/H,$$

and field extensions

$$\mathbf{C}(G/H) \hookrightarrow \mathbf{C}(G \times C) \hookrightarrow \mathbf{C}(G)((t)).$$

The valuation $\nu_{\mathcal{C}}$ on $\mathbf{C}(G/H)$ is induced by the natural valuation ord_t on $\mathbf{C}(G)((t))$. But ord_t induces a divisorial valuation on $\mathbf{C}(G \times C)$, so $\nu_{\mathcal{C}}$ is also divisorial. $\square$

Proposition 4.1.17 (Orbits induce simple valuations)**.** *For each G_∞-orbit in $\mathcal{L}(G/H)$, the associated valuation $\nu_{\mathcal{C}}$ is simple.*

Proof. From Proposition 4.1.16, we can find a complete and smooth equivariant embedding X and an invariant smooth divisor D on X such that $\nu_{\mathcal{C}} = q(\nu_{\mathcal{C}}) \cdot \mathrm{val}_D$. Let λ be the generic point of $\mathcal{C}$, which we can interpret as a morphism

$$\lambda : \operatorname{Spec} K[\![t]\!] \longrightarrow X,$$

where K is the residue field of λ as a point of X_∞. Let 0 denote the closed point in $\operatorname{Spec} K[\![t]\!]$. By construction, D is the center of the valuation $\nu_{\mathcal{C}}$ in X, and its generic point is $\lambda(0)$.

Let F be the projection of $\mathcal{C}$ on X:

$$F = \{\gamma(0) \mid \gamma \in \mathcal{C}\}.$$

The generic point of F is also $\lambda(0)$, so F is a dense subset in D. Moreover, since $\mathcal{C}$ is homogeneous with respect to the action of G_∞, the projection F is a G-orbit. This gives a dense orbit inside of D and proves that $\nu_\mathcal{C}$ is simple. □

Using the previous propositions, we get a map

$$\mathcal{L}\left(G/H\right)/G_\infty \quad \longrightarrow \quad \mathcal{V}_s(G/H).$$

The next proposition shows that this is surjective.

Proposition 4.1.18 (Every simple valuation is associated to some orbit)**.** *Let ν be a simple invariant divisorial valuation of G/H. Then there exists a G_∞-orbit $\mathcal{C}$ in $\mathcal{L}\left(G/H\right)$ such that $\nu = \nu_\mathcal{C}$.*

Proof. We can find an equivariant embedding X and an invariant divisor D such that $\nu = q \cdot \mathrm{val}_D$, where q is some positive number. We can also assume both X and D are smooth, and we pick a point x in the dense orbit of D (which exists, since the valuation ν is simple). Now we can find a smooth curve C mapping to X and intersecting D at x transversally. Let λ be the arc in X_∞ going though x corresponding to C. Since the orbit of x is dense in D, we have a dominant morphism

$$G \times C \longrightarrow X,$$

and field extensions

$$\mathbf{C}(X) \hookrightarrow \mathbf{C}(G \times C) \hookrightarrow \mathbf{C}(G)((t)).$$

The valuation ord_t in $\mathbf{C}(G)((t))$ induces both val_D and $\nu_{G_\infty \cdot \lambda}$, and we obtain the equality $q \cdot \mathrm{val}_D = \nu_{G_\infty \cdot \lambda(t^q)}$. $\square$

One would like to show that our surjection $\mathcal{L}(G/H)/G_\infty \to \mathcal{V}_s(G/H)$ is injective. But a technical problem arises: if two G_∞-orbits are contained in the same G_∞^+-orbit, the corresponding valuations are equal. I do not know an example where a G_∞^+-orbit contains more than one G_∞-orbit, but I also do not know how to prove that this cannot happen.

The last step in the proof of Theorem 4.1.11 is to show that the map $\mathcal{L}(G/H)/G_\infty^+ \to \mathcal{V}_s(G/H)$ is injective.

Proof of Theorem 4.1.11. From Propositions 4.1.16, 4.1.17 and 4.1.18, the map introduced in Definition 4.1.13 gives a surjection $\mathcal{L}(G/H)/G_\infty \to \mathcal{V}_s(G/H)$. This map factors through $\mathcal{L}(G/H)/G_\infty^+$, and it only remains to show that the resulting map is injective.

Pick a simple invariant divisorial valuation $\nu \in \mathcal{V}_s(G/H)$ and consider X, C and λ as in the proof of Proposition 4.1.18. Let $\lambda' \in \mathcal{L}(G/H)$ be a loop inducing the valuation ν. Since the map $G \times C \to X$ is smooth, we can write λ' as a product $\lambda' = \mu \cdot \gamma$, where μ is an arc in G_∞ and γ is in C_∞. In fact, γ must be of the form $\gamma(t) = \lambda(t^q\,\alpha(t))$, where α is an invertible power series. Hence λ' is in the G_∞^+-orbit of $\lambda(t^q)$, as desired. $\square$

4.2 Orbit posets and spherical varieties

In the previous section we saw that orbits in the arc space of a quasi-homogeneous variety are essentially classified by simple invariant divisorial valuations. These spaces of valuations have been computed in the literature in many specific examples, so it seems reasonable that an analysis similar to our study of generic determinantal varieties could be carried out in a more general setting.

From now on we will focus only in one aspect of the study of the arc space: understanding the dominance relation among orbits. We start by making this precise.

Definition 4.2.1 (Orbit poset)**.** Let X be an equivariant embedding of a homogeneous space G/H. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two fat G^+_∞-orbits in the arc space X_∞. We say that $\mathcal{C}_1$ *dominates* $\mathcal{C}_2$ if $\mathcal{C}_2$ is contained in the closure of $\mathcal{C}_1$, and denote it by $\mathcal{C}_1 \geq \mathcal{C}_2$. The set of fat orbits with the dominance relation is called the *orbit poset* of X_∞.

Problem 1. *Let X be an equivariant embedding of a homogeneous space G/H. Determine the structure of the orbit poset of X_∞.*

As mentioned in the introduction to this chapter, with this level of generality the problem just posed seems too hard. For example, consider an equivariant embedding X whose boundary $\partial X = X \setminus (G/H)$ has a well defined quotient variety $\partial X/G$. Then every divisorial valuation of $\partial X/G$ whose center is a point induces a simple invariant valuation of X, and therefore an orbit in X_∞. Hence understanding the orbit poset of X_∞ is at least as hard as understanding the dominance relation among all maximal divisorial sets in $(\partial X/G)_\infty$.

Of course, this is not a problem if $\partial X/G$ is zero dimensional. This motivates for us the introduction of spherical varieties.

Definition 4.2.2. Let G be a reductive group and H a closed subgroup of G. The homogeneous space G/H is said to be *spherical* if every equivariant embedding of G/H contains finitely many G-orbits. An equivariant embedding of a spherical homogeneous space is known as a *spherical variety*.

Spherical varieties have been studied extensively in the literature (Luna and Vust, 1983; Brion et al., 1986; Knop, 1991; Brion, 1993). They include toric varieties and generic determinantal varieties, as well as many other interesting spaces: flag varieties, symmetric varieties, horospherical varieties. The full theory of spherical varieties is beyond the scope of this monograph, but we include here the basic setup, as it will be useful to draw analogies between results obtained for different types of varieties.

We will not include proofs for many of the results in this section; in this cases we refer the reader to (Knop, 1991) and (Brion et al., 1986). We start with the basic characterization of spherical varieties.

Theorem 4.2.3 ((Knop, 1991) or (Brion et al., 1986, §0.2))**.** *Let G be a reductive group and H a closed subgroup. Then the following are equivalent:*

1. *G/H is a spherical homogeneous space.*

2. *A Borel subgroup of G has an open orbit in G/H.*

3. G/H is multiplicity free, i.e. for every irreducible G-module V and any character χ of H,

$$\dim_{\mathbf{C}} \{ \; v \in V \;\mid\; h\,v = \chi(h)\,v \quad \forall\, h \in H \} \leq 1.$$

The presence of a dense Borel orbit in the homogeneous space is what allows us to determine the invariant valuations, and this leads to a classification of all possible equivariant embeddings.

The theory starts by associating four objects to every spherical homogeneous space G/H: a lattice of "monomials" M; a lattice N, dual to M; a convex cone $\mathcal{V} \subset N$, the "cone of valuations"; and a set $\mathcal{D}$, the "set of colors", with a map $\varrho : \mathcal{D} \to N$.

Before defining these objects in general, we can explain what they are in the more familiar cases of toric varieties and generic determinantal varieties. In the case of a toric variety, $G = (\mathbf{C}^*)^n$ is a torus and $H = 1$ is trivial. M is the usual lattice of monomials in n variables

$$M = \{ \; x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \;\mid\; a_i \in \mathbf{Z} \; \},$$

and N is the dual of M, the lattice of one-parameter subgroups of $(\mathbf{C}^*)^n$. The cone of valuations $\mathcal{V}$ is the whole lattice, $\mathcal{V} = N$, and the set of colors is empty, $\mathcal{D} = \emptyset$.

For the case of generic determinantal varieties, assume for simplicity that we consider square matrices of size $r \times r$. Then the group G is $\mathrm{GL}_r \times \mathrm{GL}_r$, acting on the space of $r \times r$ matrices via change of basis

$$(g_1, g_2) \cdot A = g_1\, A\, g_2^{-1}.$$

The subgroup H is the stabilizer of the identity matrix, i.e. $H = \mathrm{GL}_r$ embedded diagonally inside of G. Consider the following polynomials in the entries of a generic $r \times r$ matrix:

$$A = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1s} \\ x_{21} & x_{22} & \cdots & x_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ x_{r1} & x_{r2} & \cdots & x_{rs} \end{pmatrix}, \qquad \begin{array}{ll} \Delta_1 = \det A & y_1 = \Delta_1/\Delta_2 \\ \vdots & \vdots \\ \Delta_{r-1} = x_{r,r}x_{r-1,r-1} - x_{r-1,r}x_{r,r-1} & y_{r-1} = \Delta_{r-1}/\Delta_r \\ \Delta_r = x_{rr} & y_r = \Delta_r \end{array}$$

Then M is the lattice of monomials in $y_1, \ldots, y_r$,

$$M = \{\ y_1^{a_1} y_2^{a_2} \cdots y_r^{a_r} \ \mid\ a_i \in \mathbf{Z}\ \},$$

and N is the dual of M, the lattice of one-parameter subgroups of the diagonal torus in GL_r,

$$N = \{\ \mathrm{diag}(t^{\lambda_1}, t^{\lambda_2}, \ldots t^{\lambda_r}) \ \mid\ \lambda_i \in \mathbf{Z}\ \}.$$

The cone of valuations is

$$\mathcal{V} = \{ \operatorname{diag}(t^{\lambda_1}, t^{\lambda_2}, \dots t^{\lambda_r}) \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \} \subset N.$$

The set of colors is $\mathcal{D} = \{D_2, \dots, D_r\}$, where D_i is the divisor in G/H given by $\Delta_i = 0$. Each divisor in $\mathcal{D}$ defines a valuation of G/H, which by restriction to M defines an element in N. In this case we get an inclusion of $\mathcal{D}$ in N:

$$\begin{array}{rcl} \varrho : \mathcal{D} & \longrightarrow & N \\ D_2 & \longmapsto & \operatorname{diag}(t^{-1}, t, 1, \dots, 1, 1), \\ D_3 & \longmapsto & \operatorname{diag}(1, t^{-1}, t, \dots, 1, 1), \\ & \vdots & \\ D_r & \longmapsto & \operatorname{diag}(1, 1, 1, \dots, t^{-1}, t). \end{array}$$

For an arbitrary spherical homogeneous space G/H, we define M, N, $\mathcal{V}$ and $\mathcal{D}$ as follows. Let B be a Borel subgroup of G such that BH/H is dense in G/H, and let T be the maximal torus in B. We denote by $\mathbf{C}(G/H)^{(B)}$ the abelian group of B-semi-invariant rational functions of G/H:

$$\mathbf{C}(G/H)^{(B)} = \{ f \in \mathbf{C}(G/H)^* \mid \exists \chi \text{ character of } T \text{ s.t. } g \cdot f = \chi(g) f \quad \forall g \in B \}.$$

Then M is defined as the subgroup of functions in $\mathbf{C}(G/H)^{(B)}$ that have value 1 at the base point H/H:

$$M = \{ f \in \mathbf{C}(G/H)^{(B)} \mid f(H/H) = 1 \}.$$

One can identify M with the quotient $\mathbf{C}(G/H)^{(B)}/\mathbf{C}^*$, which is isomorphic to the image of the natural map from $\mathbf{C}(G/H)^{(B)}$ to the character group of T. This shows that M is a finitely generated lattice (Knop, 1991, §1).

We define N as the dual lattice of M. Since M is included in the character lattice of T, we know that the group of one-parameter subgroups of T surjects onto a finite index sublattice of N. In other words, possibly after multiplying by a positive integer, every element in N can be represented by a one-parameter subgroup of T.

We define $\mathcal{V}$ as the set of invariant divisorial valuations of $\mathbf{C}(G/H)$. It can be shown that the valuations in $\mathcal{V}$ are all simple (Luna and Vust, 1983, §8.10), and that they are determined by their values on the functions in M (Knop, 1991, Cor. 1.8). We get an inclusion $\mathcal{V} \subset N$, and $\mathcal{V}$ gets identified with the set of integral points in a convex cone in $N \otimes \mathbf{Q}$.

$\mathcal{D}$ is the set of B-invariant prime divisors of G/H, and its elements are called the *colors* of G/H. Since BH/H is dense in G/H, colors correspond to codimension one components of the complement $(G/H) \setminus (BH/H)$, so $\mathcal{D}$ is a finite set. To each color we can associate a B-invariant valuation of G/H, and hence an element of N by restriction. This gives a map $\varrho : \mathcal{D} \to N$, which needs not be injective.

In order to recover our description of M, N, $\mathcal{V}$ and $\mathcal{D}$ in the cases of toric varieties, one only needs to realize that in this case $G = B = T$. For generic determinantal varieties, the Borel subgroup that we chose is the one containing pairs of matrices of the following shape:

$$\begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & * \end{pmatrix}, \begin{pmatrix} * & 0 & \cdots & 0 \\ * & * & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{pmatrix}$$

Once M, N, $\mathcal{V}$ and $\mathcal{D}$ have been determined, one can classify all equivariant embeddings.

Definition 4.2.4. An equivariant embedding of a homogeneous space G/H is said to be *simple* if it contains a unique closed orbit.

Remark 4.2.5. Toric varieties are simple if and only if they are affine. Generic determinantal varieties are all simple, as the unique closed orbit in this case is the one that contains the zero matrix.

To each simple equivariant embedding $X \supset G/H$ with closed orbit Y, one associates a subcone $\mathcal{V}_X \subset \mathcal{V}$ and a set of colors $\mathcal{D}_X \subset \mathcal{D}$ in the following way. $\mathcal{V}_X$ is the set of G-invariant valuations of G/H having a center on X. The elements in $\mathcal{D}_X$ are those colors whose closure contains Y (recall that the colors are the B-invariant prime divisors of G/H). The pair $(\mathcal{V}_X, \mathcal{D}_X)$ is known as the *colored cone* associated to the simple equivariant embedding X.

Theorem 4.2.6 ((Luna and Vust, 1983, Prop. in §8.3) or (Knop, 1991, Thm. 2.3))**.** *Let G/H be a spherical homogeneous space. A simple equivariant embedding X of G/H is determined, up to isomorphism of equivariant embeddings, by its associated colored cone $(\mathcal{V}_X, \mathcal{D}_X)$.*

It is possible to characterize which pairs $(\mathcal{W}, \mathcal{F})$ appear as colored cones of simple equivariant embeddings, and by introducing the notion of *colored fan* one can classify all embeddings, not necessarily simple (Knop, 1991, §3). But for our purposes, knowing how one associates a colored cone to a simple equivariant embedding will be enough.

In the case of toric varieties, the theorem just says that an affine toric variety X is determined by $(\sigma_X, \emptyset)$, where σ_X is the cone associated to X.

In the generic determinantal case, when $X = \mathbf{A}^{r^2}$ is the space of $r \times r$ matrices, $\mathcal{D}_X = \mathcal{D}$ and $\mathcal{V}_X$ is given by

$$\mathcal{V}_X = \{ \ \mathrm{diag}(t^{\lambda_1}, t^{\lambda_2}, \ldots t^{\lambda_r}) \ \mid \ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0 \ \}.$$

Note that $\mathcal{V}_X$ can be identified with the space Λ_r of partitions of length at most r, an object that played a crucial role in Chapter 3. The previous theorem says that X is the only equivariant embedding of GL_r whose associated colored cone is $(\Lambda_r, \mathcal{D})$.

Given a simple equivariant embedding X, Theorem 4.1.11 says that $\mathcal{V}_X$ indexes fat G_∞^+ orbits in X_∞. In the cases of toric varieties and generic determinantal varieties, one is able to understand $\mathcal{V}_X$ not only as a cone inside of N, but also as a poset, with the order induced by the dominance relation among orbits in the arc space. This leads to the following refined version of the problem we posed at the beginning of the section.

Problem 2. *Let G/H be a spherical homogeneous space, and let X be a simple equivariant embedding of G/H with associated colored cone $(\mathcal{V}_X, \mathcal{D}_X)$. Describe the partial order on $\mathcal{V}_X$ induced by the relation of dominance among orbits in the arc space X_∞.*

There is another interesting order in the cone of valuations: given two valuations ν_1 and ν_2 defined over an affine variety X, one says that $\nu_1 \leq_X \nu_2$ if $\nu_1(f) \leq \nu_2(f)$ for all regular functions $f \in \mathcal{O}_X$. Given two cylinders $\mathcal{C}_1$ and $\mathcal{C}_2$ in the arc space X_∞, it is clear that $\nu_{\mathcal{C}_1} \leq_X \nu_{\mathcal{C}_2}$ whenever $\mathcal{C}_1$ dominates $\mathcal{C}_2$. Using this idea and the following structural theorem for simple spherical varieties, one easily obtains a first approximation to Problem 2.

Theorem 4.2.7 ((Knop, 1991, Thm. 2.1 and Thm. 2.5))**.** *Let X be a simple spherical variety with closed orbit Y and associated colored cone $(\mathcal{V}_X, \mathcal{D}_X)$, and let $\sigma_X \subset N$ be the cone generated by $\mathcal{V}_X$ and $\varrho\mathcal{D}_X$. We define the* big cell *X_B as the set of points $x \in X$ such that the closure of $B \cdot x$ contains Y. Then:*

1. *X_B is a B-stable open affine subset of X,*
2. *$X = G \cdot X_B$,*
3. *$\mathbf{C}[X_B]^{(B)}/\mathbf{C}^* = \sigma_X^\vee \cap M$.*

Proposition 4.2.8. *Let X be a simple spherical variety with associated colored cone $(\mathcal{V}_X, \mathcal{D}_X)$, and let σ_X be as in Theorem 4.2.7. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two fat orbits in the arc space X_∞, and assume that $\mathcal{C}_1$ dominates $\mathcal{C}_2$. Let ν_1, ν_2 be the associated valuations in $\mathcal{V}_X$. Then:*

$$\nu_2 - \nu_1 \in \sigma_X.$$

Proof. Since $\mathcal{C}_2 \subset \overline{\mathcal{C}_1}$, we know that $\nu_2 \geq \nu_1$ as valuations over X. In particular, if we restrict these valuations to X_B we get

$$\nu_2(f) - \nu_1(f) \geq 0 \qquad \text{for all } f \in \mathbf{C}[X_B]^{(B)}.$$

From part 3 of Proposition 4.2.7, we see that $\nu_2 - \nu_1$ is non-negative on $\sigma_X^\vee$. Hence $\nu_2 - \nu_1 \in \sigma_X^{\vee\vee} = \sigma_X$, as required. □

For generic determinantal varieties, Proposition 4.2.8 is equivalent to Proposition 3.3.4. To see this, recall that in this case $\mathcal{V}_X$ is the set of partitions of length at most r,

$$\mathcal{V}_X = \{\ (\lambda_1, \dots, \lambda_r)\ \mid\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0\ \},$$

and that $\varrho\mathcal{D}_X$ contains the vectors

$$(-1, 1, 0, \dots, 0, 0), \quad (0, -1, 1, \dots, 0, 0), \quad \dots \quad (0, 0, 0, \dots, -1, 1).$$

Then it is immediate to check that σ_X, the cone generated by $\mathcal{V}_X$ and $\varrho\mathcal{D}_X$, is given by

$$\sigma_X = \{ \ (\lambda_1, \ldots, \lambda_r) \ | \ \lambda_r + \lambda_{r-1} + \cdots + \lambda_{r-i} \geq 0 \ \ \forall i \ \}.$$

If $\nu_1 = (\lambda_1, \ldots, \lambda_r)$ and $\nu_2 = (\mu_1, \ldots, \mu_r)$, the condition $\nu_2 - \nu_1 \in \sigma_X$ is equivalent to

$$\lambda_r + \cdots + \lambda_{r-i} \leq \mu_r + \cdots + \mu_{r-i} \qquad \forall i.$$

In Chapter 3 we expressed this condition by saying that ν_1 is subordinate to ν_2. Therefore Proposition 4.2.8 just says that domination of orbits implies subordination of the corresponding partitions.

Note that in Theorem 3.3.11 we proved the converse to Proposition 4.2.8 for generic determinantal varieties.

In the rest of the chapter we give answers to Problem 2 in various specific examples. In all of them we will see that the converse to Proposition 4.2.8 holds, but we do not know whether this is true for all simple spherical varieties.

4.3 Toric varieties

In this section we analyze the dominance relation among orbits in the arc space of an affine toric variety. The result is due to Ishii (Ishii, 2004), but we include here a slightly modified proof, avoiding resolution of singularities.

We follow the notation and conventions of (Fulton, 1993). In particular, $T = (C^*)^n$ denotes a torus, M is the lattice of characters of T, and N is the lattice of one-parameter subgroups. Given an affine toric variety X, one naturally associates to it a cone $\sigma_X = \mathcal{V}_X \subset N$ which contains those one-parameter subgroups having a limit in X. The dual cone $S_X = \sigma_X^\vee \subset M$ is in bijection with the monomials in $\mathcal{O}_X$. More precisely, the ring $\mathcal{O}_X$ equals $\mathbf{C}[S_X]$, the semi-group algebra generated by S_X. Given an element $u \in S_X$, we denote by $x^u \in \mathbf{C}[S_X]$ the corresponding monomial, and by $\langle u, v \rangle$ the natural pairing of u with some element in N.

In the language of Section 4.2, the cone of valuations $\mathcal{V}$ is the whole lattice N. From Theorem 4.1.11, this means that $N = \mathcal{V} = \mathcal{L}(T)/T_\infty^+$ is the set of T_∞^+-orbits in $\mathcal{L}(T)$, and that σ_X parametrizes fat T_∞^+-orbits in the arc space X_∞ of a toric variety X. An easy direct computation shows that $\mathcal{L}(T)/T_\infty = \mathcal{L}(T)/T_\infty^+$. In other words, T_∞-orbits are also T_∞^+-orbits, both in the loop space $\mathcal{L}(T)$ and in the arc space X_∞ of any toric variety X.

Theorem 4.3.1 ((Ishii, 2004, Prop. 4.8))**.** *Let X be an affine toric variety, and $\nu_1, \nu_2 \in \sigma_X$ two one parameter subgroups. Let $\mathcal{C}_1, \mathcal{C}_2 \subset X_\infty$ be the associated T_∞-orbits in the arc space. Then*

$$\mathcal{C}_1 \textit{ dominates } \mathcal{C}_2 \qquad \Longleftrightarrow \qquad \nu_2 - \nu_1 \in \sigma_X.$$

Proof. From Proposition 4.2.8 it is enough to show the sufficient condition. Notice that affine toric varieties are monoids in the category of schemes. Indeed, if $S_X = \sigma_X^\vee \subset M$ is the semigroup of monomials associated to X, there is a natural comultiplication in $\mathcal{O}_X = \mathbf{C}[S_X]$,

$$\Delta : \mathbf{C}[S_X] \to \mathbf{C}[S_X] \otimes \mathbf{C}[S_X], \qquad \Delta(x) = x \otimes x \quad \forall x \in S_X,$$

and a natural counit

$$\epsilon : \mathbf{C}[S_X] \to \mathbf{C}, \qquad \epsilon(x) = 1 \quad \forall x \in S_X,$$

The composition

$$\mathbf{C}[S_X] \xrightarrow{\Delta} \mathbf{C}[S_X] \otimes \mathbf{C}[S_X] \xrightarrow{\subset} \mathbf{C}[S_X] \otimes \mathbf{C}[M]$$

gives the action of the tours on X, and X is a group only when $X = T$, as $\mathbf{C}[S_X]$ admits an antipode compatible with Δ and ϵ if and only if S_X is a group.

The structure of monoid transfers to the arc space. In particular, multiplication by an arc is a well-defined continuous endomorphism of X_∞.

Consider $\nu_0 = \nu_2 - \nu_1$ and its associated T_∞-orbit $\mathcal{C}_0$ in the arc space X_∞. Since X_∞ is irreducible and T_∞ is open in X_∞, we know that $\mathcal{C}_0$ is in the closure of T_∞. More explicitly, the one-parameter subgroup ν_0 extends uniquely to an arc in X_∞, and the composition

$$\mathbf{C}[S_X] \xrightarrow{\nu_0} \mathbf{C}[\![t]\!] \xrightarrow{t \to s+t} \mathbf{C}[\![s,t]\!], \qquad x^u \mapsto (s+t)^{\langle u, \nu_0 \rangle},$$

gives a wedge whose generic arc is in T_∞ and specializes to an arc in $\mathcal{C}_0$ when $s = 0$.

As X is a commutative monoid, multiplication by an arc sends T_∞-orbits to T_∞-orbits. In particular, from the fact that $\nu_1 + 0 = \nu_1$ and $\nu_1 + \nu_0 = \nu_2$, we see that multiplication by ν_1 sends T_∞ to $\mathcal{C}_1$ and $\mathcal{C}_0$ to $\mathcal{C}_2$. Since multiplication by ν_1 is continuous and $\mathcal{C}_0$ is in the closure of T_∞, it follows that $\mathcal{C}_2$ is in the closure of $\mathcal{C}_1$. More explicitly, the map

$$w : \mathbf{C}[S_X] \to \mathbf{C}[\![s,t]\!], \qquad w(x^u) = t^{\langle u,v_1\rangle}\,(s+t)^{\langle u,v_0\rangle} = st^{\langle u,v_1\rangle} + t^{\langle u,v_2\rangle},$$

gives a wedge whose generic arc is in $\mathcal{C}_1$ but specializes to an arc in $\mathcal{C}_2$ when $s = 0$. $\square$

4.4 Skew-symmetric matrices

The techniques of Chapter 3 apply also to the study of arc spaces of varieties of skew-symmetric matrices. In this section we focus on the analysis of the dominance relation among orbits in the arc space.

Throughout the section, $X = \mathbf{A}^{2r^2-r}$ will denote the space of skew-symmetric matrices of size $2r \times 2r$. The group $G = \mathrm{GL}_{2r}$ acts on X via

$$g \cdot A \quad = \quad g^{-T} A g^{-1}.$$

The action of G on X has exactly $r+1$ orbits, corresponding to the varieties of skew-symmetric matrices of rank $2i$ for $i \in \{0, \ldots, r\}$. This is just a consequence of standard Gaussian elimination applied to elements of X: after coordinated row and column operations, any skew-symmetric matrix can be transformed into a block-diagonal matrix of the form

$$\begin{pmatrix} a_1 \cdot J & 0 & \dots & 0 \\ 0 & a_2 \cdot J & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_r \cdot J \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (a_1, \dots, a_r) = (0, \dots, 0, 1, \dots, 1).$$

This also shows that the determinant of a skew-symmetric matrix of even size $2i \times 2i$ is always a square; the square root of this determinant is known as the Pfaffian of the skew-symmetric matrix. Note that determinants of skew-symmetric matrices of odd size $(2i+1) \times (2i+1)$ are zero. We denote by $D^k \subset X$ the space of skew-symmetric matrices of rank at most $2k$. It can be shown that D^k is an algebraic subset of X whose defining ideal is generated by the Pfaffians of $2(k+1) \times 2(k+1)$-submatrices.

Matrices of rank $2r$ give a dense orbit, $X \setminus D^{r-1}$, which is isomorphic to $\mathrm{GL}_{2r} / \mathrm{Sp}_{2r}$. Write $H = \mathrm{Sp}_{2r}$, and let $B \subset G$ be the Borel subgroup of lower triangular matrices. Then $BH/H \subset X$ is the set of skew-symmetric matrices that can be transformed into a block-diagonal form without permuting rows or columns, and starting the elimination with the lower-right block. Let Δ_i be the Pfaffian of the lower-right $2(r-i+1) \times 2(r-i+1)$-submatrix. Then the divisor $\Delta_1 \Delta_2 \cdots \Delta_r = 0$ is the complement of BH/H in X, and we see that $X \supset G/H$ is a spherical variety. It is in fact a simple spherical variety, with unique closed orbit $D^0 = \{0\}$.

Since $\Delta_1 \cdots \Delta_r = 0$ is the complement of BH/H in X, the lattice $M = \mathbf{C}(G/H)^{(B)}/\mathbf{C}^*$ is generated by $\Delta_1, \ldots, \Delta_r$. It will be convenient to consider the regular functions

$$y_1 = \Delta_1/\Delta_2, \qquad \ldots \qquad y_{r-1} = \Delta_{r-1}/\Delta_r, \qquad y_r = \Delta_r,$$

and write

$$M = \{ \; y_1^{a_1} y_2^{a_2} \cdots y_r^{a_r} \;\mid\; a_i \in \mathbf{Z} \; \}.$$

Let N be the dual lattice to M. Elements in N can be represented by loops in $\mathcal{L}\,(G/H)$ of the form

$$\begin{pmatrix} t^{\lambda_1} \cdot J & 0 & \ldots & 0 \\ 0 & t^{\lambda_2} \cdot J & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & t^{\lambda_r} \cdot J \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \lambda_i \in \mathbf{Z}.$$

Using Gaussian elimination, we see that any loop in $\mathcal{L}\,(G/H)$ can be transformed to a loop in N via coordinated row and column operations using coefficients in $\mathbf{C}[\![t]\!]$. In other words, any G_∞-orbit contains a representative in N. But there is some ambiguity in this representation: many loops in N belong to the same orbit. If we restrict ourselves to row and column operations from B_∞ we eliminate this redundancy. We see that the set of orbits in $\mathcal{L}\,(G/H)$ is indexed by the set

$$\mathcal{V} = \{ \; (\lambda_1, \lambda_2, \ldots, \lambda_r) \;\mid\; \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \; \} \subset N.$$

Not all orbits in $\mathcal{L}(G/H)$ are contained in X_∞, we need the exponents of t to be positive. Fat G_∞-orbits in X_∞ are parametrized by

$$\mathcal{V}_X = \{ \ (\lambda_1, \lambda_2, \ldots, \lambda_r) \ | \ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0 \ \} \simeq \Lambda_r.$$

Note that $\mathcal{V}_X$ is isomorphic to the set of partitions of length at most r. As in the case of generic determinantal varieties, if we allow λ_i to take the value infinity we recover all G_∞-orbits, not only the fat ones. Also notice that in this case G_∞-orbits are G_∞^+-orbits.

The colors of G/H correspond to B-invariant prime divisors in G/H, i.e. prime divisors in the complement of BH/H in G/H. In this case they are $\mathcal{D} = \{E_2, \ldots, E_r\}$, where E_i is the divisor defined by $\Delta_i = 0$ (note that Δ_1 is a unit in $\mathbf{C}[G/H]$, so it does not define a color even though it is B-invariant). Recall that the set $\mathcal{D}_X$ of colors associated to X is formed by divisors containing the closed orbit $D^0 = \{0\}$. In this case $\mathcal{D}_X = \mathcal{D}$. Each color defines a valuation, which by restriction to M defines an element of N. In this case the map $\varrho : \mathcal{D} \to N$ is

$$E_2 \mapsto (-1, 1, 0, \ldots, 0, 0), \quad E_3 \mapsto (0, -1, 1, \ldots, 0, 0), \quad \ldots, \quad E_r \mapsto (0, 0, 0, \ldots, -1, 1).$$

The cone $\sigma_X \subset N$ is generated by $\mathcal{V}_X$ and $\varrho\mathcal{D}_X$. As in the case of generic determinantal varieties, we get

$$\sigma_X = \{ \ (\lambda_1, \ldots, \lambda_r) \ | \ \lambda_r + \lambda_{r-1} + \cdots + \lambda_{r-i} \geq 0 \ \ \forall i \ \}.$$

On the set $\mathcal{V}_X = \Lambda_r$ of partitions of length at most r, the cone σ_X induces the partial order of subordination:

$$\mu - \lambda \in \sigma_X \quad \Longleftrightarrow \quad \lambda_r + \cdots + \lambda_{r-i} \leq \mu_r + \cdots + \mu_{r-i} \quad \forall i \quad \Longleftrightarrow \quad \lambda \preceq \mu.$$

Theorem 4.4.1. *Let X be the space of skew-symmetric matrices of size $2r \times 2r$. Then the orbit poset of X_∞ is anti-isomorphic to the partition poset $(\Lambda_r, \preceq)$.*

Proof. Given partitions $\lambda, \mu \in \Lambda_r$, denote by $\mathcal{C}_\lambda$ and $\mathcal{C}_\mu$ the corresponding orbits in X_∞. From Propositions 4.2.8 and 3.3.10, it is enough to show that the closure of $\mathcal{C}_\lambda$ contains $\mathcal{C}_\mu$ when $\lambda \preceq \mu$ and μ is a cover of λ.

Assume first that μ is a cover of λ of type 2 (recall Definition 3.3.7). Consider the following wedge in X:

$$w = \begin{pmatrix} (st^{\lambda_1} + t^{\mu_1}) \cdot J & 0 & \dots & 0 \\ 0 & t^{\lambda_2} \cdot J & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{\lambda_r} \cdot J \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The generic arc of w is in $\mathcal{C}_\lambda$, but it specializes to an arc in $\mathcal{C}_\mu$ when $s = 0$.

Assume now that μ is a hyper-cover of λ of type 3 or of type 4, and let $i < j$ be the indices such that $\mu_i = \lambda_i - 1$, $\mu_j = \lambda_j + 1$ and $\mu_k = \lambda_k$ for $k \neq i, j$. Consider the following wedge:

$$w = \begin{pmatrix} t^{\lambda_1}\cdot J &&&&&&&&& \\ & \ddots &&&&&&&& \\ && t^{\lambda_{i-1}}\cdot J &&&&&&& \\ &&& \alpha & 0 & \cdots & 0 & \beta && \\ &&& 0 & t^{\lambda_{i+1}}\cdot J & \cdots & 0 & 0 && \\ &&& \vdots & \vdots & \ddots & \vdots & \vdots && \\ &&& 0 & 0 & \cdots & t^{\lambda_{j-1}}\cdot J & 0 && \\ &&& -\beta & 0 & \cdots & 0 & \delta && \\ &&&&&&&& t^{\lambda_{j+1}}\cdot J & \\ &&&&&&&&& \ddots \\ &&&&&&&&&& t^{\lambda_r}\cdot J \end{pmatrix},$$

where J is as above and

$$\Omega = \begin{pmatrix} \alpha & \beta \\ -\beta & \delta \end{pmatrix} = \begin{pmatrix} 0 & st^{\lambda_i} + t^{\lambda_i - 1} & t^{\lambda_i - 1} & 0 \\ -st^{\lambda_i} - t^{\lambda_i - 1} & 0 & 0 & st^{\lambda_j} \\ -t^{\lambda_i - 1} & 0 & 0 & st^{\lambda_j} + t^{\lambda_j + 1} \\ 0 & -st^{\lambda_j} & -st^{\lambda_j} - t^{\lambda_j + 1} & 0 \end{pmatrix}.$$

Note that the ideal of 2×2 Pfaffians of Ω is

$$\left(st^{\lambda_i} + t^{\lambda_i - 1}, \quad t^{\lambda_i - 1}, \quad st^{\lambda_j}, \quad st^{\lambda_j} + t^{\lambda_j + 1} \right) \quad = \quad \begin{cases} (t^{\lambda_j + 1}) & \text{if } s = 0, \\ (t^{\lambda_j}) & \text{if } s \text{ is a unit,} \end{cases}$$

and the 4×4 Pfaffian of Ω is $t^{\lambda_i + \lambda_j}\left(1 + st + s^2\right)$. Hence the generic arc of w is in $\mathcal{C}_\lambda$, but it specializes to an arc in $\mathcal{C}_\mu$ when $s = 0$. $\square$

4.5 Toroidal embeddings of symmetric spaces

Symmetric spaces are a particularly well understood class of spherical homogeneous spaces. In this section we study the dominance relation among orbits in the arc space for embeddings of symmetric spaces. We will give a complete answer for the class of *toroidal* embeddings.

Definition 4.5.1. Let G be a semi-simple simple-connected algebraic group, and consider an involution $\sigma \neq \mathrm{id}_G$. The subgroup of fixed points of σ will be denoted by G^σ, and its normalizer by $N_G(G^\sigma)$. Let H be a closed subgroup of G such that

$$G^\sigma \subset H \subset N_G(G^\sigma).$$

The homogeneous space G/H is known as an (*algebraic*) *symmetric space*. It can be shown that symmetric spaces are spherical (Goodman and Wallach, 1998, §12.3).

We will use the theory of embeddings of symmetric spaces as developed by de Concini and Procesi in (de Concini and Procesi, 1983) and (de Concini and Procesi, 1985). In this approach, one does not study arbitrary embeddings, we need to restrict ourselves to *toroidal embeddings*.

Definition 4.5.2. Let G/H be a spherical homogeneous space, and X a simple equivariant embedding of G/H. Let $(\mathcal{V}_X, \mathcal{D}_X)$ be the associated colored cone. We say that X is *toroidal* if $\mathcal{D}_X = \emptyset$. For an arbitrary embedding X, we way that X is *toroidal* if no color contains a closed orbit.

We will need the following facts, for which we refer the reader to (Goodman and Wallach, 1998; de Concini and Procesi, 1983; de Concini and Procesi, 1985). Fix a symmetric space G/H, and let M, N, and $\mathcal{V}$ be as in Section 4.2. Then there exists a root system Φ on N such that $\mathcal{V}$ is one of its Weyl chambers. Let W be the Weyl group associated to this root system. Let $S \subset G$ be a maximal σ-anisotropic torus of G, i.e. $\sigma(s) = s^{-1}$ for all $s \in S$ and S is maximal with respect to this property. Consider $\overline{S} = S/S \cap H$. One can identify M with the character lattice of $\overline{S}$.

Let now X be a toroidal embedding of G/H, and let $x = H/H$ be its base point. It is shown in (de Concini and Procesi, 1985) that the closure X_S of $\overline{S} \cdot x$ in X is a toric variety with respect to $\overline{S}$, on which W acts. In fact, they prove that the fan associated to X_S is W-invariant and each of its cones is contained in some Weyl chamber for Φ. We will call a fan in N verifying these two properties Φ-*admissible*. The map $X \mapsto X_S$ gives an equivalence between the category of toroidal embeddings of G/H and the category of toric varieties with respect to $\overline{S}$ whose fan is Φ-admissible. In particular, since the Weyl chamber decomposition is itself a Φ-admissible fan, it has an associated toroidal embedding of G/H, known as the *wonderful embedding*. A fan is Φ-admissible if and only if it maps to the Weyl chamber decomposition, therefore an equivariant embedding is toroidal if an only if it maps to the wonderful embedding.

The previous description of the toroidal embeddings of symmetric spaces is all we need to describe completely the dominance relation among orbits in the arc space.

Theorem 4.5.3. *Let X be a simple toroidal embedding of a symmetric space G/H, and let $(\mathcal{V}_X, \emptyset)$ be the associated colored cone. Let $\mathcal{C}_1, \mathcal{C}_2 \subset X_\infty$ be two fat G_∞^+-orbits, and consider the associated valuations $\nu_1, \nu_2 \in \mathcal{V}_X$. Then*

$$\mathcal{C}_1 \textit{ dominates } \mathcal{C}_2 \qquad \Longleftrightarrow \qquad \nu_2 - \nu_1 \in \mathcal{V}_X.$$

Proof. From Proposition 4.2.8 we only need to prove the sufficient condition. Assume that $\nu_2 - \nu_1 \in \mathcal{V}_X$. Let X_S be the toric variety associated to X (see the discussion preceding the theorem). The fan of X_S is obtained from $\mathcal{V}_X$ by letting the Weyl group W act. In particular $\mathcal{V}_X$ is a member of this fan, and we can consider the affine toric variety $Z \subset X_S$ associated to it. Let $\mathcal{C}_1^S$ and $\mathcal{C}_2^S$ the $\overline{S}_\infty$-orbits in Z_∞ associated to ν_1 and ν_2. By construction, the inclusion $Z_\infty \to X_\infty$ sends $\mathcal{C}_i^S$ into $\mathcal{C}_i$ in an $\overline{S}_\infty$-equivariant way. By 4.3.1, we have that $\mathcal{C}_2^S \subset \overline{\mathcal{C}_1^S}$. Therefore $\mathcal{C}_2 \cap \overline{\mathcal{C}_1} \neq \emptyset$ and the result follows. $\square$

Remark 4.5.4. The statement of the previous theorem can be refined to obtain results beyond the toroidal case. Let X be a simple embedding of a symmetric space with colored cone $(\mathcal{V}_X, \mathcal{D}_X)$. Then there is a simple toroidal embedding X' with colored cone $(\mathcal{V}_X, \emptyset)$ and a map of embeddings $X' \to X$. Let $\mathcal{C}_1$, $\mathcal{C}_2$ be orbits in $\mathcal{L}(G/H)$ with associated valuations ν_1, ν_2. If $\nu_2 - \nu_1 \in \mathcal{V}_X$, the previous theorem shows that $\mathcal{C}_1$ dominates $\mathcal{C}_2$ in X'_∞. But since the map $X'_\infty \to X_\infty$ is continuous, $\mathcal{C}_1$ also dominates $\mathcal{C}_2$ in X_∞.

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10/2008 – University of Michigan

Algebraic Geometry and Commutative Algebra (Hartshorne's 70th birthday)
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Arcs on Determinantal Varieties

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We study the structure of the arc space and the jet schemes of generic determinantal varieties. Via the use of group actions, we are able to compute the number of irreducible components of all the jet schemes, find formulas for log canonical thresholds, and compute some motivic volumes.

We also study extensions of our results for determinantal varieties to more general quasi-homogeneous spaces, with focus on spherical varieties. We obtain good descriptions for the space of skew-symmetric matrices, and for toroidal embeddings of symmetric spaces.